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AN INVESTIGATION OF FLUID FLOW IN A WATER LUBRICATED BEARING

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AN INVESTIGATION OF FLUID FLOW IN A WATER LUBRICATED BEARING

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Submitted in partial fulfillment
of the requirements
for the degree of
MASTER OF SCIENCE
in MECHANICAL ENGINEERING

United States Naval Postgraduate School
Annapolis, Maryland
1949

This work is accepted as fulfilling
the thesis requirements for the degree of
MASTER OF SCIENCE
in
MECHANICAL ENGINEERING

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United States Naval Postgraduate School

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PREFACE

The U. S. Naval Engineering Experiment Station, Annapolis, Maryland, is at present conducting tests on water lubricated bearings. High pressures, high water temperatures, high angular speeds, and fine clearances are involved in these tests.

The authors first became interested in the problem of fluid flow through water lubricated bearings during an informal discussion with Mr. Watt V. Smith, Superintendent of Bearings Project, U. S. Naval Engineering Experiment Station, who is in charge of the tests. The Engineering Experiment Station has thus far largely confined its investigation to the problems of load carrying capacity and erosion effects. The fluid flow through these bearings appeared to be an interesting problem and the authors offered their services to the EES for an analysis of the flow. Mr. Smith made the full test apparatus available to the authors for their investigation.

With the hope of assisting the Engineering Experiment Station in the evaluation of test data, the authors chose the task of finding suitable formulae and graphical methods for determining the rate of flow through a water lubricated bearing from known pressure and temperature conditions in the bearing.

The authors wish to express their deep appreciation

to all who have aided in this investigation. They are especially grateful to Associate Professor George H. Lee, U. S. Naval Postgraduate School, for his continued guidance and for his careful review of the completed manuscript.

The authors are also grateful to personnel of the U. S. Naval Engineering Experiment Station for their whole-hearted cooperation in providing the test apparatus and for their assistance in conducting this investigation. Mr. Watt V. Smith suggested the investigation and gave the authors many helpful hints as the investigation proceeded. Further, an acknowledgment would be incomplete without mention of Mr. M. Levinsohn and Mr. W. J. Vitelozzi, who were tireless in their efforts to keep the bearing apparatus in operating condition and also to assist the authors in conducting their test runs.

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TABLE OF SYMBOLS AND ABBREVIATIONS

A - area

a - journal radius

b - bearing inner radius

C - diametral clearance 2(b - a)

c - bearing shell outer radius

D - journal diameter

E - Modulus of elasticity in tension and compression

EES - U. S. Naval Engineering Experiment Station

e - eccentricity ratio (ratio of the distance between the centers of the journal and bearing to the radial clearance).

F - force

g - acceleration due to gravity

h - radial clearance (b - a)

K - Taylor's criterion of instability

L - length

M - mass rate of flow

m - hydraulic radius

N - angular journal speed (RPM)

P - pressure difference between two points

P'' - bearing load per unit of projected area

P_{∞} - equivalent bearing load due to side leakage

p_x - pressure at any point x

Q - volume rate of flow

q - volume rate of flow due to pumping action

R - Reynolds number

r - radius

T - temperature difference between two points

t_x - temperature at any point x

U - average fluid velocity in axial direction

V - displacement in radial direction

W - displacement in axial direction

Z - axis parallel to shaft

$\alpha_{(1,2,3)}$ coefficient of linear thermal expansions of shaft, bearing metal and bearing backing, respectively

β - constant in Suzuki's equation for turbulent flow

γ - Poissons ratio

$\Delta(\ )$ - small finite difference of quantity in parantheses

δ - unit increase in volume

ϵ_r - unit deformation in radial direction

ϵ_t - unit deformation in tangential direction

ϵ_z - unit deformation in axial direction

μ - absolute viscosity

ν - kinematic viscosity

ρ - density

σ_r - normal stress in the radial direction

σ_t - normal stress in the tangential direction

σ_z - normal stress in the axial direction

τ - shear stress

ϕ - angle

ω - angular journal speed

ω_c - critical angular velocity

CHAPTER I

INTRODUCTION

Water lubricated bearings are used in a number of applications, the most important in Naval Engineering being the stern tube and strut bearings for ship propulsion machinery. These bearings, as well as the majority of water lubricated bearings, employ large clearances and comparatively slow journal speeds.

The analysis of the above mentioned bearing is quite different from the problem posed by a full journal bearing with small clearance, high shaft speed and with lubrication by water at high temperature and pressure. Such a bearing might find application in modern high-speed steam turbines and pumps where the use of water as the bearing lubricant would simplify the present elaborate gland seal system and make impossible the introduction of oil into the feed system from turbine or pump bearings. The analysis of axial flow of water through such a bearing, subject to light loads, is the subject of this investigation.

Analyses of "end leakage" or axial flow in oil lubricated bearings are given in various lubrication publications. These analyses, as far as the authors have been able to determine, confine their treatment to stable or laminar flow, and are therefore inadequate for our problem where large pressure drops through the bearing and high speeds of journal rotation will necessarily result in turbulent flow.

Extensive investigations have been made of water flow between rotating concentric cylinders. Neglecting eccentricity, our bearing problem reduces to the particular case of water flow between two concentric cylinders, the inner cylinder rotating and the outer cylinder fixed. While an analysis of this case has been made by a number of investigators, the tests at the EES are with smaller clearances and higher speeds of journal rotation than any results known to the authors.

The problem of stable flow between rotating concentric cylinders is quite involved and there is considerable disagreement among investigators as to what criterion to use to determine stability. The most widely accepted criterion for stability of flow between rotating concentric cylinders with no axial flow was developed mathematically and verified experimentally by Taylor (18)* Since Taylor first published his criterion in 1923, a number of investigators have verified his results. For the specific case of only the inner cylinder or journal rotating, Taylor finds stable flow existing below a definite and determinable speed of rotation, while above this speed unstable or turbulent flow can be expected. Goldstein (8) verified Taylor's results and further found that the same criterion was satisfactory for axial flow with low values of Reynolds Number, R. Fage (7) extended Taylor's and Goldstein's investigations for

*Numbers in brackets refer to bibliography values of R up to 680. Fage's results are applicable to our bearing problem and are reproduced in figure 4.

The criterion evolved by Taylor and extended by Goldstein and Fage appears satisfactory for determining the conditions which must be satisfied for stable flow in a water lubricated bearing; but, of course, it furnishes no solution for the rate of flow through the bearing. Empirical formulas as well as graphical solutions based upon dynamical similarity exist for turbulent flow between concentric cylinders with the inner cylinder rotating. By taking into account the effect of eccentricity caused by the load on the bearing, we may apply the above mentioned formulas and graphical solutions to our bearing problem.

In the following chapters the authors will present both empirical equations and graphical solutions for flow through water lubricated bearings. In this investigation, however, the authors find that the solution is adequately represented by graphical plots such as shown in figures 3 and 11. Furthermore, graphical solutions may be used for both turbulent and laminar flow.

In analyzing a problem of this type it may be assumed that at sufficiently slow speeds of journal rotation and with moderate pressure drops through the bearing, laminar flow will result. In the classical equation for laminar flow between concentric cylinders, the rate of flow varies directly as the clearance cubed and the pressure gradient axially along the cylinder, and inversely as the viscosity of the flowing fluid. Therefore, accurate knowledge of the running clearance of the bearing is essential for predicting the flow. The temperature conditions throughout the bearing must be known because of their effect upon the viscosity of the water and, to a lesser degree, upon the clearance in the bearing. Accurate knowledge of entrance and exit pressures in the bearing is also essential. These pressures affect the bearing clearance and, most important of all, affect the rate of flow.

The EES bearing poses a number of problems which have not ordinarily been present in previous experiments to determine the flow between rotating concentric cylinders.

The first and most difficult problem is accurate determination of the running clearance of the bearing. As the mean diametrical clearance as measured at 68° F. is only .0038 inches, any small variation in this clearance results in relatively large changes of flow. There is a difference in coefficients of thermal expansion of the journal and the materials composing the bearing, which results in a change of clearance with change in temperature conditions in the bearing. At high temperatures there will necessarily be a temperature gradient across the bearing which further complicates the determination of the clearance. Further, in order to withstand the high pressures involved and at the same time prevent leakage, the design and assembly of the EES bearing required certain end restraints upon the individual bearings in order to retain them in place. The magnitude of these end restraints are unknown and their effect upon the clearance cannot be determined.

Then too, there is a light load on the EES bearing which makes necessary the consideration of the effects of eccentricity. As the eccentricity of any particular bearing will vary with changes of load, journal speed, and pressure drop across the bearing, the exact solution of this problem is extremely difficult.

Under conditions of low rates of flow and high journal speeds, the temperature of the water increases as it flows through the bearing. The resulting change in viscosity may have an appreciable effect upon the flow through the bearing.

The problems posed by the EES bearing will exist in varying degrees in any operating water-lubricated bearing. Detailed analysis of the EES bearing is made in Chapter IV.

CHAPTER II

ORIGIN OF DATA

The test bearing consists of four bearings on a vertical test shaft of constant diameter as shown in figure 1. The shaft is necked down to 3/8" diameter before extending through the upper and lower stuffing boxes in the bearing housing. The two central bearings are in a common bearing shell. All bearings have 1/8" radius fillets on each end of the bearing metal. The central bearings can be loaded in a radial direction by means of a spring. The shaft is rotated by means of a belt driven pulley; the pulley being mounted on the shaft above the bearing. Combination sleeve and thrust bearings are installed on either side of the drive pulley to carry the weight of the shaft and to prevent any radial thrust from being transmitted to the test bearing. For convenience the bearings have been numbered from one to four from top to bottom. The bearing housing is supported at its base on a rigid stand. A photograph of the test apparatus is shown in figure 2.

The finish of the journal varies from 2.0 to 3.25 micro inches (RMS) and the finish of the bearing metal surfaces varies from 32 to 50 micro inches (RMS).

Each of the four bearings may have different metal temperatures. The temperatures of bearings 1 and 4 are recorded as t_1 and t_6, the temperature of the middle two bearings as t_{16}. These temperatures are measured by thermocouples. Temperatures t_2 and t_4 are taken at the top and bottom of the bearing inner casing and temperatures t_{11}, t_{12}, and t_{10} are taken on the outside surfaces of the bearing housing near the top, middle, and bottom respectively. All thermocouple positions are indicated in figure 1.

All temperature measurements are taken by means of iron-constantan wire thermocouples and are recorded by a continuous-reading recorder. Ambient temperature is recorded as t_{15}. The temperatures obtained by these thermocouples are correct to plus or minus 2 deg. F. in absolute temperature and plus or minus 1 deg. F. for temperature differences.

All pressures are determined by means of Bourdon tube pressure gauges. These gauges are connected to the water inlet pipe, both discharge pipes, and to the space behind bearings 2 and 3. The gauge locations and pressure ranges are also indicated in figure 1. The error of the pressure gauge readings does not exceed 1½% of the maximum reading to which the scale is graduated; except that the error in reading the graduations between 33-1/3% and 66-2/3" of the maximum scale reading does not exceed 1/2% of the maximum scale reading to which the scale is graduated.

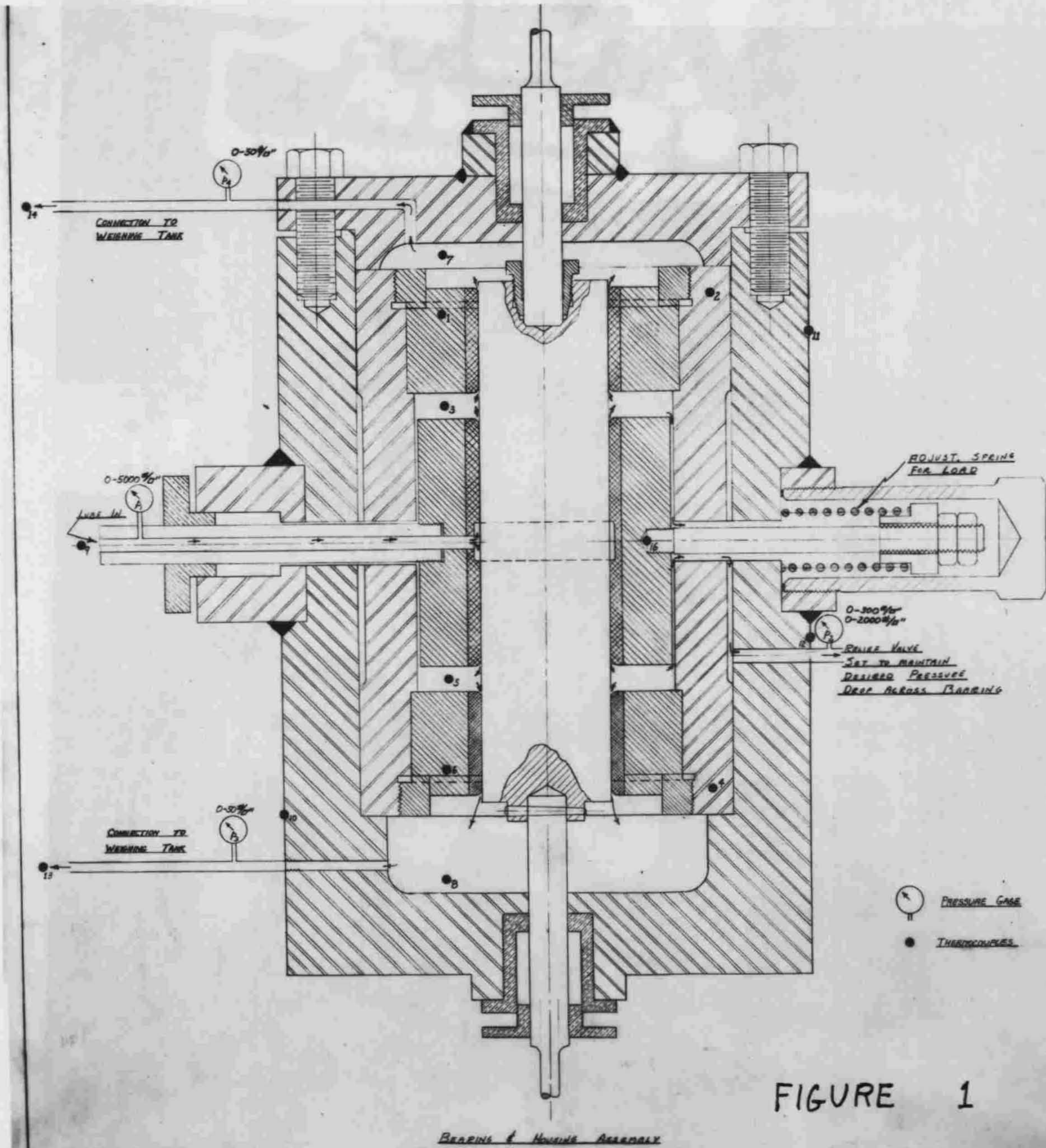

FIGURE 1

BEARING & HOUSING ASSEMBLY
15000 RPM VERTICAL BEARING TEST -
WATER LUBRICATED

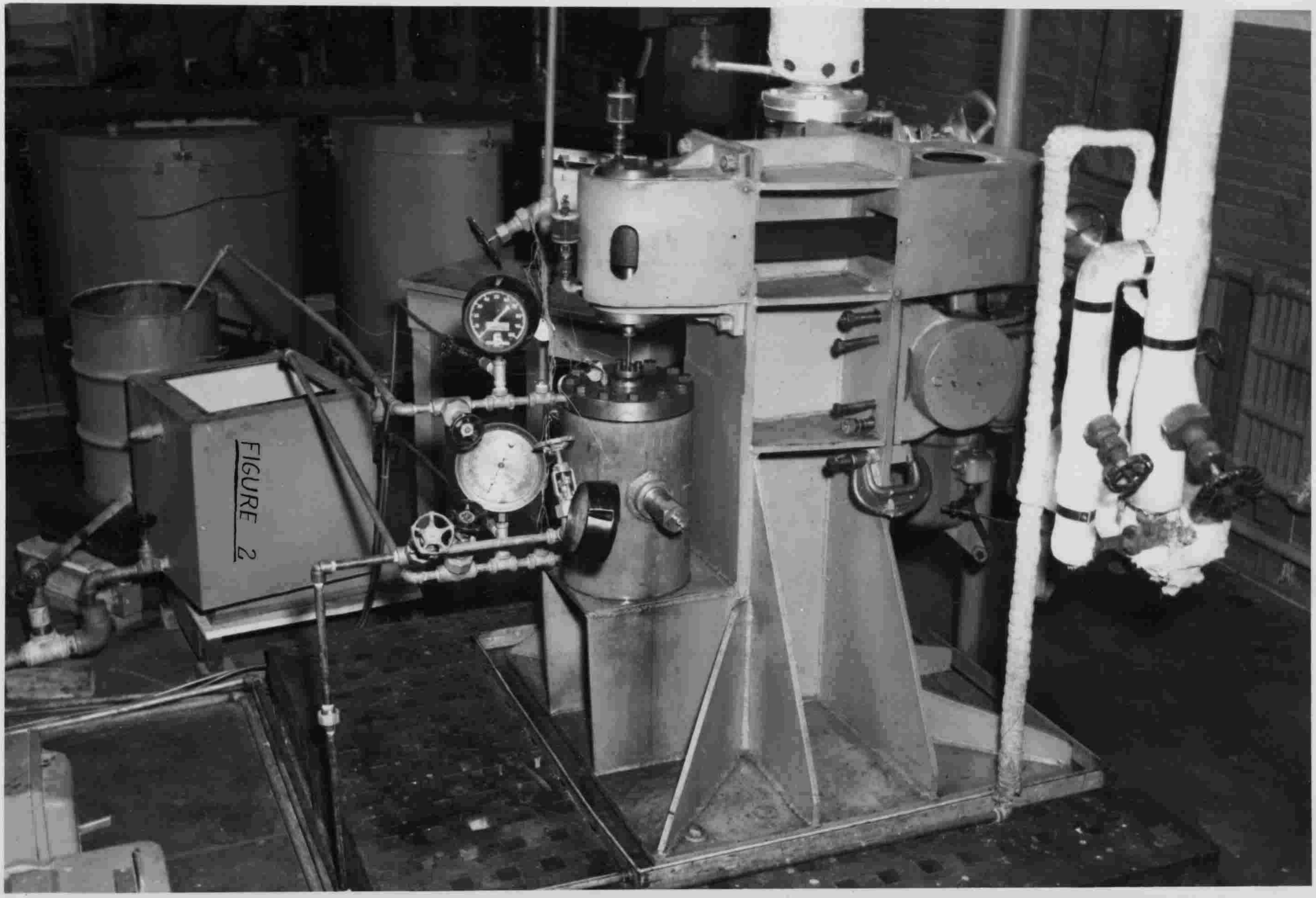

FIGURE 2

The mean diametral bearing clearances for each of the four bearings are designated as C_1, C_2, C_3, and C_4 respectively. All reference measurements are taken under standard conditions at 68°F. The accuracy of the measurement of the journal and bearing diameters is correct within plus or minus 2 ten thousandths of an inch.

Water is admitted through a pipe to the circular groove between bearings 2 and 3 at pressure p_1 and temperature t_9, flows axially through bearings 2 and 3 to the spaces between bearings 1 & 2 and 3 & 4. These spaces are interconnected by approximately 1/8" clearance in back of the bearing shell which houses bearings 2 and 3, so that the pressures will be the same, p_2; however, the temperatures need not necessarily be the same and we use t_3 and t_5 to represent the temperatures in the upper and lower spaces respectively. The flow through bearings 1 and 4 is again axial and the resulting pressures and temperatures after flow through the bearings are p_4, t_7 and p_3, t_8 at the top and bottom respectively. The discharge from each end is led to separate collecting tanks located on platform scales. Here the quantities of flow are weighed and temperatures are taken as t_{14} and t_{13} for flow from top and bottom bearings respectively. To insure that the discharge pressures p_3 and p_4 are the same a mercury manometer is connected across the discharge lines. Flow from either line can be throttled by means of a globe valve to equalize the mercury columns of the manometer.

The materials of construction with approximate percentage compositions are shown below:

Shaft	U.S.I.S. Symbol 440c alloy steel (1.0 C, 17 Cr, 0.75 Mo)
Bearing Material-	Bearium #10 (70 Cu, 20 Pb, 10 Sn)
Bearing Shell-	Fed. Specs. 46S1-M Steel (0.31C,0.75 Mn, 0.25 Si, 0.35 Cu, 0.25 Ni)
Inner Casing-	COMP M Bronze (88 Cu, 6.5 Sn, 4 Zn, 1.5 Pb)
Outer Housing-	Fed.Specs. 4952-AN Forged Steel (0.45 C, 2.75 Ni)

CHAPTER III

THEORETICAL SOLUTIONS

In this chapter the authors will present general solutions for the rate of axial flow through a water lubricated bearing. Their solutions will include both laminar and turbulent flow and they will also discuss the various criteria for stability of flow.

As previously stated in Chapter I, laminar flow may be expected in a water lubricated bearing under certain conditions. The equation for laminar flow between two concentric cylinders is given by Lamb (12) as

$$Q = \frac{\pi P}{8\mu L}\left[b^4 - a^4 - \frac{(b^2 - a^2)^2}{\ln \frac{b}{a}}\right] \qquad (1)$$

Where

Q is the volume rate of flow;

P is the axial pressure difference through the bearing;

μ is the absolute viscosity;

L is the length of the bearing;

b is the inner radius of the bearing;

a is the radius of the journal.

Equation (1) may be put into a more convenient form by substituting for b its equivalent, $a + h$, and expanding to*

$$Q = \frac{\pi P a h^3}{6\mu L}\left[1 + \frac{1}{2}\left(\frac{h}{a}\right) + \frac{1}{60}\left(\frac{h}{a}\right)^2 + \frac{1}{120}\left(\frac{h}{a}\right)^3 + \cdots\right]. \qquad (2)$$

---

* See Appendix (I) for mathematical derivation.

When the radial clearance is small compared to the radius, equation (1) becomes

$$Q = \frac{\pi P a h^3}{6 \mu L} \qquad (3)$$

Equation (3) above is also the formula for viscous flow between two parallel flat plates.

The diametral clearance C is frequently used in bearing calculations. Substituting C for 2h and D for 2r, equation (3) becomes

$$Q = \frac{\pi P D C^3}{96 \mu L} \qquad (4)$$

Further, the eccentricity of the journal must be considered. With the journal stationary and for laminar flow the eccentricity may increase the quantity of flow for a given pressure difference by as much as 150 percent over that with no eccentricity. The effect of eccentricity may be obtained by reference to Orloff (14). Equation (4) then becomes

$$Q = \frac{\pi P D C^3}{96 \mu L} \left(1 + 1.5 e^2\right) \qquad (5)$$

where e is the eccentricity ratio and has a maximum vâlue of unity when the journal is hard over against the bearing.

The mean axial velocity U may be obtained from equation (5) above by dividing Q, the axial flow, by the cross-sectional area of the flow path, $\frac{\pi D C}{2}$*. Formula (5) then becomes

---

*Approximate area, where radial clearance is small compared to the journal radius.

$$U = \frac{PC^2}{48 \mu L} (1 + 1.5e^2). \qquad (6)$$

Equation (5) does not consider the effect of rotation with an eccentric journal. Orloff gives a formula for additional end leakage due to the action of pressure developed in the loaded zone. With approximations this formula reduces to

$$q = \frac{1.2 \text{ X } 10^{-10} \, ND^3 \, (C/D)^{2.2}}{\left(\frac{\mu \omega}{P''}\right)^{0.6}} \left[D/L(1 + 0.6D/L \right] \qquad (7)$$

Where

q is volume rate of flow (ft^3/sec);

N is angular journal speed (RPM);

D is journal diameter (inches);

ω is angular journal speed (rad/sec);

P'' is projected bearing pressure.

An examination of equations (5) and (7) will show that for light loads and reasonably large pressure drops through the bearing the additional flow as computed by equation (7) becomes negligible small. Therefore, equation (5) is sufficient for effecting a solution of laminar flow in the EES bearing.

Thus far entry and exit losses have been neglected. Where the pressure drop used in effecting a solution of the flow problem is obtained from the difference of pressures measured in the high and low pressure chambers, at the entrance and exit of the bearing respectively, the effect of these losses is to decrease the measured pressure drop. Cornish (3), in his experiments with flow between concentric cylinders with the inner cylinder rotating, found that the speed of rotation had very little effect upon the entrance and exit losses and that these losses approximated $\frac{U^2}{2g}$. Cornish attributed his low value of these losses to slight "bell mouthing" of the entrance.

The bearings used by the authors were rounded at the entrances and exits by a 1/8 inch fillet. Their apparatus provided no means for determining entry and exit losses. The authors found, however, that the actual rates of flow in the laminar flow region were generally as great or greater than the calculated flow as determined by neglecting these losses. Furthermore, assuming these losses are approximately $\frac{U^2}{2g}$, then a correction of $\frac{U^2}{2g}\rho$ may be made to the measured pressure drop across the bearing. This correction becomes appreciable only at very high velocities.

No such simple formulas as previously given for laminar flow will suffice for turbulent flow in a water lubricated bearing. As is usually done in turbulent flow problems, the principle of dynamical similarity may be applied. If we follow the methods used by Cornish we obtain these dimensionless parameters and their relations:

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$$\frac{Pm^2}{\mu LU} = f_1\left(\frac{mU}{\nu}, \frac{r}{m}, \frac{m\omega}{U}\right) \qquad (8)$$

$$\frac{m\omega_c}{U} = f_2\left(\frac{mU}{\nu}, \frac{r}{m}\right) \qquad (9)$$

where

m is the hydraulic radius;

ν is the kinematic viscosity;

r is the mean radius;

ω_c is the critical angular velocity.

Cornish was able to reduce the number of parameters by multiplication of individual parameters, thus

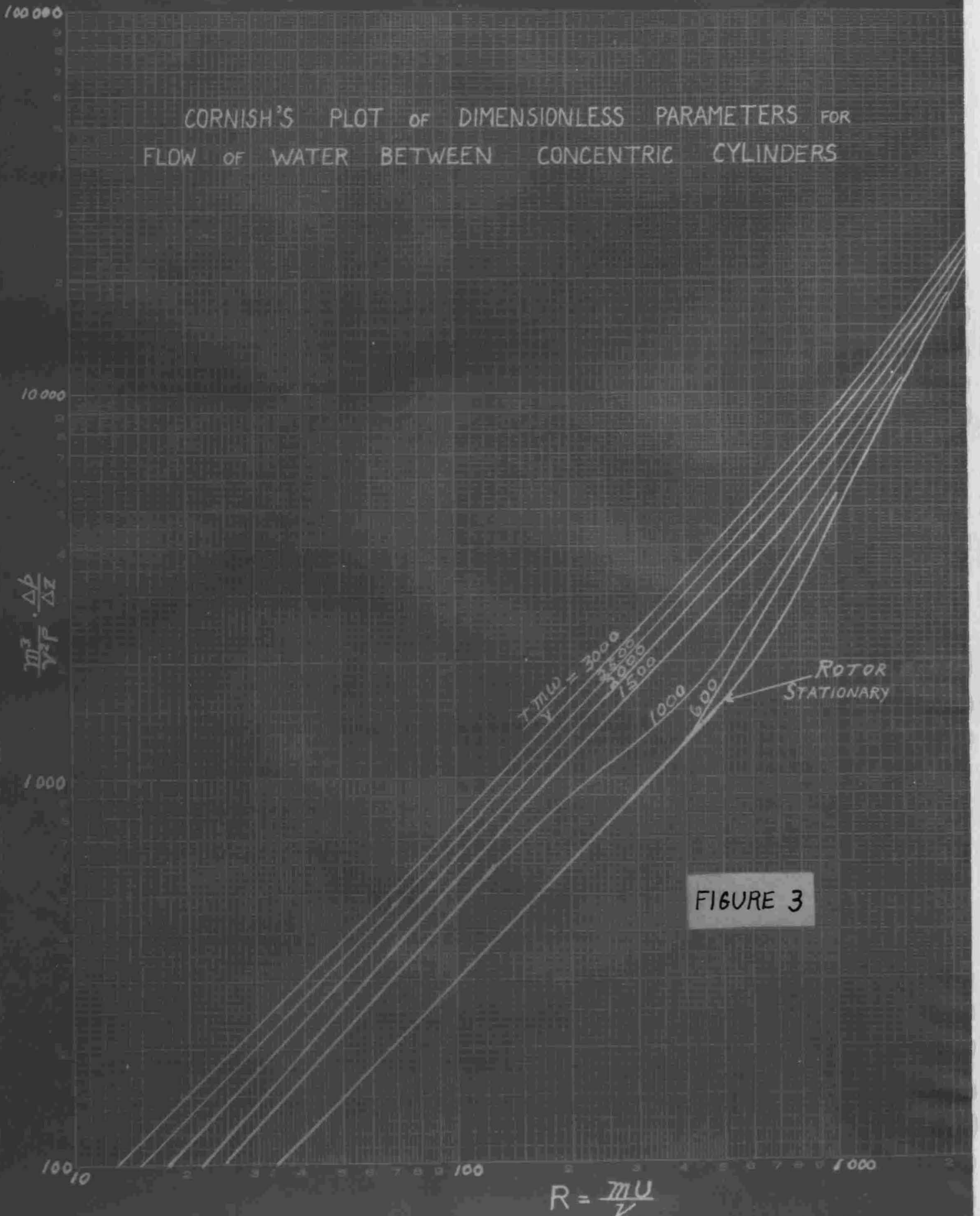
$$\frac{mU}{\nu} \cdot \frac{r}{m} \cdot \frac{m\omega}{U} = \frac{rm\omega}{\nu} \qquad (10)$$

$$\frac{Pm^2}{\mu LU} \cdot \frac{mU}{\nu} = \frac{m^3p}{\nu^2\rho L} \qquad (11)$$

where ρ is the density of the fluid.

Cornish then plotted $\frac{m^3P}{\nu^2\rho L}$ against R for various values of $\frac{rm\omega}{\nu}$ to obtain his figure 7, as reproduced in figure 3. Figure 3 is a very useful family of curves and provides a means for quickly obtaining a solution for the flow through a water lubricated bearing. This solution neglects the effects of eccentricity. Cornish used values of r/m ranging from 341 to 624 to obtain his curves and states that valid results can be expected over this range. A large number of bearing problems might be expected to fall within this range.

A brief mention should be made of Cornish's apparatus. He experimented with five sets of apparatus, the general



CORNISH'S PLOT OF DIMENSIONLESS PARAMETERS FOR FLOW OF WATER BETWEEN CONCENTRIC CYLINDERS

FIGURE 3

construction being a smooth brass cylinder rotating inside a hollow cylinder also of brass and finished smooth. Rotation was accomplished by means of a D.C. motor, giving a range of speeds from 0 to 2000 RPM. The mean radius r varied from 6 to 10 centimeters and the length of cylinders varied from 15 to 28 centimeters. His values of $\frac{r}{m}$ varied from 138 to 624.

The values of r/m used by the authors were much higher than those used by Cornish, varying from 1319 to 1369. However, Cornish's work appears more closely allied with the investigation of the authors than the work of other experimenters. It is of considerable interest to note Cornish's curves converge rapidly in the highly turbulent region and are thus essentially independent of shaft rotation in this area. This is important in this investigation for the high journal speeds and large pressure drops involved necessarily result in highly turbulent flow.

It should be noted here that the results obtained by Cornish do not entirely agree with results of later investigators, namely Goldstein (8) and Fage (7). Cornish differs in his methods of determining critical angular velocity for the breakdown of stable flow. The criteria for stability used by the various investigators will be discussed shortly; but regardless of his determination of the beginning of instability, the major portion of Cornish's curves should be reliable over the range of r/m from 341 to 624. While the authors used values of r/m varying from 1319 to 1369, they obtained fair agreement with Cornish as is shown by comparison of figures 3 and 11.

It must be emphasized that figure 3 neglects the effects of eccentricity and that eccentricity is negligible in figure 11 except the curve for $\frac{rm\omega}{\nu} = 0$. For this curve with the rotor stationary the apparent eccentricity ratio as calculated by formula (5) varied approximately from 0.7 at R equal 100 to 0.3 at R equal 320. With light loads on the bearing and at high journal speeds the eccentricity is negligibly small and figures 3 and 11 will give adequate solutions to applicable bearing problems. For values of r/m near 500 Cornish's plot as shown in figure 3 is recommended for bearing problems. With values of r/m near 1345, the plot shown in figure 11 may be used.

A number of formulas for turbulent flow exist which are applicable to the flow through a water lubricated bearing. Davies and White (4) give a general formula for turbulent flow in a pipe of any form with smooth boundaries, which when applied to this problem gives

$$\frac{m^2 P}{\mu LU} = 0.0283 \left(\frac{mU}{\nu} \right)^{3/4} \qquad (12)$$

Suzuki (16) derived his formula for flow between concentric cylinders, the inner cylinder rotating, by working from Karman's law of velocity distribution.

For our problem his formula reduces to

$$\frac{m^2P}{\mu LU} = 0.0283 \left(\frac{mU}{\nu}\right)^{3/4} \left[\frac{1}{2}\left\{\left(1+0.629\,\frac{r^2\omega^2}{U^2}\right)^{3/8} + \left(1+0.629\,\frac{\beta^2 r^2\omega^2}{U^2}\right)^{3/8}\right\}\right] . \quad (13)$$

Suzuki did not give a value for the constant β which appears to vary with clearance. Cornish found close agreement with β equal to 0.8 for values of R from 1000 upward. The authors, in their investigation, found insufficient agreement with Suzuki's formula to merit quoting any value of the constant β .

The problem of stability of flow of a viscous fluid between two infinitely long coaxial rotating cylinders with no axial flow was first solved mathematically and verified experimentally by Taylor. Taylor's formula for our case of the inner cylinder only rotating is

$$\frac{\pi^4 \nu^2 \;(a+b)}{2\omega_c^2 \;\; h^3 \; a^2} = K \qquad (14)$$

Where $K = 0.0571 \left(1 - 0.652\,\frac{h}{a}\right) + 0.00056 \left(1 - 0.652\,\frac{h}{a}\right)$.
Equation (14) above was rewritten by Fage in the more convenient form

$$\frac{2\omega_c \; a\,m}{\nu} \left(\frac{a+b}{m}\right)^{-\frac{1}{2}} = \frac{\pi^2}{2\sqrt{K}} . \qquad (15)$$

Therefore, for the case of no flow; i.e., R equal to zero, the speed of rotation where turbulence first sets in is easily calculated by using formula (15) above.

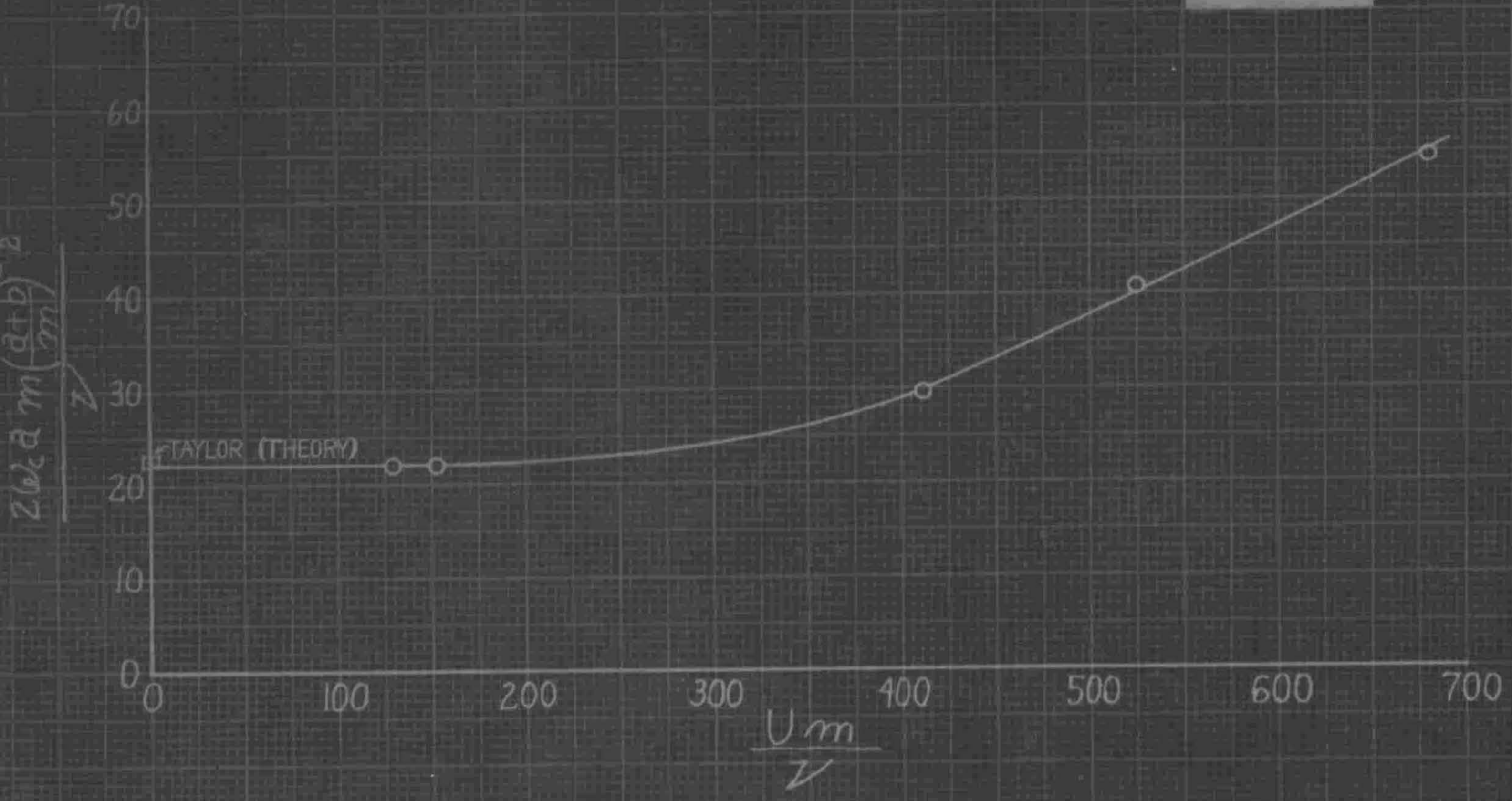
Goldstein investigated the stability of flow under the influence of a pressure gradient parallel to the axis and with the inner cylinder rotating at a uniform speed. He found Taylor's formula gave good results at low values of R; i.e., $R < 10$. Fage experimentally extended Goldstein's work, obtaining a plot of $2\omega_c am\left(\frac{a+b}{m}\right)$ versus R for values of R up to 680. Taylor's point for R equal to zero fits smoothly on Fage's curve. Fage's plot is shown in figure 4.

Cornish found $\frac{r\omega c}{U}$ essentially independent of r/m over a range of values of r/m from 341 to 624. An examination of equations (14) and (15) shows that according to Taylor's theory for no axial flow $\frac{r\omega c}{U}$ varies as $(r/m)^{\frac{1}{2}}$. Since Goldstein and Fage later found that Taylor's criterion for stability might be modified to apply to axial flow considerable doubt has been cast upon the method used by Cornish to obtain the critical rotational speed.

Fage's plot appears most useful for determining the breakdown of stability for flow through a water lubricated bearing, but it offers no means of determining the rate of flow. The authors feel that a graphical plot such as made by Cornish represents the best solution to our problem of flow through a water lubricated bearing.

FAGE'S PLOT OF CRITICAL ANGULAR VELOCITY VERSUS REYNOLDS NUMBER

FIGURE 4



CHAPTER IV

SOLUTION OF THE EES PROBLEM

The following particulars are given for the EES problem.

Measurements (68°F.)

Main bearing, length	4.75	in.
Central circumferential groove, width	.75	in.
End bearing, length	2.00	in.
Journal, diameter (mean)	2.5003	in.
Bearing, mean inside diameter, No.1	2.5041	in.
Bearing, mean inside diameter, No.2	2.5039	in.
Bearing, mean inside diameter, No.3	2.5044	in.
Bearing, mean inside diameter, No.4	2.5041	in.
Bearing metal, thickness	.250	in.
Bearing, mean outside diameter, Nos. 1 & 4	5.2493	in.
Bearing, mean outside diameter, Nos. 2 & 3	4.8750	in.

Materials of construction

See Chapter II, Origin of Data.

Physical properties of materials

	α (in./in./°F.)	E (psi)	Compressive σ_{yp} (psi)	(γ)
Journal	5.9×10^{-6}	-----	-----	---
Bearing metal	10.28×10^{-6}	10×10^{6}	12,000	.33
Bearing backing	6.51×10^{-6}	28.8×10^{6}	-----	.26

Load applied to bearings	100 lb.
Load applied to each bearing	50 lb.
Temperature, water	83-202°F.
Pressure drop across end bearings	49-1120 psi

Angular journal speeds 0 - 12,000 R.P.M.

Calculations are made for bearings 1 and 4 only. As shown in figure 1 there is a radial clearance of 1/8 inch behind bearings 2 and 3 permitting exchange of flow between the discharges of these central bearings. Due to this construction, the actual measurement of flow through these individual bearings is impossible. This design feature provides equal pressures at the entrances of bearings 1 and 4.

For laminar flow through the EES bearing the quantity of flow may be readily obtained by application of equation (5). When the flow is turbulent we must either obtain a solution from a graphical plot of dimensionless parameters or apply empirical equations of known accuracy. Inasmuch as a graphical plot of dimensionless parameters will give a solution for either laminar or turbulent flow, the authors believe the graphical method offers the best solution to the EES problem.

Sufficient data was taken and a plot of $\frac{m^3 P}{\nu^2 \rho L}$ versus R for constant values of $\frac{rm\omega}{\nu}$ is shown in Figure 11. It should be emphasized here that Figure 11 is theoretically applicable only to a bearing with an $\frac{r}{m}$ approximately the same as the EES bearing, i.e., 1345. Similarity would further require the same eccentricity ratio. Concerning $\frac{r}{m}$, the authors agree with Cornish that exact equality of $\frac{r}{m}$ for similarity is not required and therefore figure 11 may be used for a fairly wide range of values of $\frac{r}{m}$. For values of $\frac{r}{m}$ near 500

Cornish's curves as shown in figure 3 are recommended.

The solution of the EES problem may be best illustrated by calculating one of the points in figure 11. A calculation of run 6* will illustrate the method followed by the authors in obtaining points for the curves of figure 11. The foot-pound-second system will be used in the following calculations unless indicated otherwise.

The factors which must be considered in determining the mean clearance are: (1) the expansion of the bearing and shaft due to change in temperature from 68°F; (2) the pressure effects on the bearing surfaces; and (3) the increase in shaft radius due to rotation.

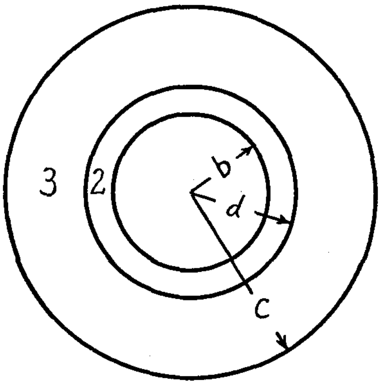


Figure 5 End View of Bearing

Considering first the expansion of the bearing due to temperature we wish to compute the displacement of the inner radius b. We may consider the bearing to be made up of two shells shown as parts 2 and 3 in figure 5 above. Part 2 is bonded to part 3. Upon heating the bearing there will be created a pressure between the bearing metal and the backing since α_2 is larger than α_3. The radial stresses in parts 2 and 3 due to the difference in coefficients of expansion are

---

*See Data Sheet and Calculated Data

$$\sigma_3 = \frac{pd}{(c - d)}$$

and

$$\sigma_2 = \frac{pd}{(d - b)}$$

where p is the pressure at the bonded surface.

The pressure p is determined from the condition that during heating both tubes have the same circumferential elongation, i.e.

$$\alpha_3(t-68) + \frac{pd}{E_3\,(c-d)} = \alpha_2(t-68) - \frac{pd}{E_2(d-b)}$$

or

$$p = \frac{(\alpha_2 - \alpha_3)(t-68)}{d\left[\frac{1}{E_3(c - d)} + \frac{1}{E_2(d - b)}\right]} .$$

Substituting for p in the expression for σ_2 gives

$$\sigma_2 = \frac{(\alpha_2 - \alpha_3)(t - 68)E_2}{1 + \frac{(d - b)}{(c - d)}\ \frac{E_2}{E_3}} .$$

Substituting numerical values in the above expression we have

$$\sigma_2 = \frac{(10.28 - 6.51)10^{-6}\ (92 - 68)10 \times 10^6}{1 + \frac{.25}{.94}\left(\frac{10 \times 10^6}{28.8 \times 10^6}\right)} = -828 \text{ psi compressive.}$$

Some stresses may exist in the parts of the bearing due to the bonding process. We have no means of measuring these stresses which are normally tensile stresses in the bearing metal. As the stress due to pressure as computed above is only 828 psi compressive and as the bonding stresses are normally tensile, it is to be expected that the combined stresses will be less than the yield strength of the bearing metal, i.e., 12,000 psi. Therefore, no plastic flow of the bearing metal is to be expected.

The unit deformation in the tangential direction is

$$\epsilon_{2t} = \alpha_2(t - 68) + \frac{\sigma_2}{E_2} .$$

Therefore

$$\Delta d = \frac{\Delta \text{circumference}}{2\pi} = \frac{2\pi d\, \epsilon_{2t}}{2\pi} = \left[\alpha_2(t-68) + \frac{\sigma_2}{E_2}\right] d .$$

Substituting numerical values gives

$$\Delta d = \left[10.28 \times 10^{-6}(92-68) - \frac{828}{10 \times 10^6}\right] 1.5 = 244 \times 10^{-6} \text{in.}$$

The unit deformation in the radial direction is

$$\epsilon_{2r} = \alpha_2(t - 68) - \gamma \frac{\sigma_{2t}}{E_2}$$

$$= 10.28 \times 10^{-6}(92-68) - .33\frac{(-828)}{10 \times 10^6} = 273 \times 10^{-6} \text{in./in.} .$$

The change in radial thickness of the bearing metal is

$$\Delta(d - b) = \epsilon_{2r}(d - b) = 273 \times 10^{-6}(.25) = 68 \times 10^{-6} \text{ in.}$$

Hence

$$\Delta b = \Delta d - \Delta(d - b)$$

$$= \left[244 - 68\right] 10^{-6} = 176 \times 10^{-6} \text{ inches}$$

which is the change in b due to change in temperature.

The effect of the pressure on the bearing shell will be considered next. Due to the construction of bearings 1 and 4, the pressure p_2 will be present on the outer surface of the bearing shell since the gaskets to prevent water leakage in back of the bearings are on the down stream ends of the bearings.

Formulas (28) and (32) of Appendix II will now be employed to determine the radial displacement, Δb, of the bearing inner radius.

The actual bearing is composed of bearing metal .25 in. thick with an E of 10×10^6 psi and the bearing backing is .94 in. thick with an E of 28.8×10^6 psi so that negligible error will be introduced by assuming a homogeneous steel bearing for calculating Δb due to pressure effects only.

Since the bearing temperatures t_1 and t_6 and the outer housing temperatures t_{10}, t_{11}, and t_{12} correspond, the temperature gradient aspects of formulas (28) and (32) will be omitted. Equation (32) then reduces to

$$\epsilon_z = -\frac{P_{end}}{E} - \frac{2\gamma\left(b^2 p_i - c^2 p_o\right)}{E\left(c^2 - b^2\right)}$$

where P_{end} is our pressure, p_2, acting on each end of the bearing and p_i may be taken as the arithmetic mean of p_2 and p_4. Then upon substitution we have

$$\epsilon_z = -\frac{255}{28.8 \times 10^6} - \frac{2 \times .26 \left((1.25)^2\, 153 - (2.44)^2\, 255\right)}{28.8 \times 10^6 \left((2.44)^2 - (1.25)^2\right)}$$

$$= -3.6 \times 10^{-6} \text{ in./in.}$$

Substituting the above value of ϵ_z in equation (28) of Appendix II gives

$$\begin{aligned}\Delta b = (1+\gamma) \Bigg[& \frac{r(1-2\gamma)}{E} \left(\frac{b^2 p_i - c^2 p_o}{c^2 - b^2}\right) + \frac{b^2\, c^2}{r(c^2 - b^2)} \left(\frac{P_i - p_o}{E}\right) \\ & - \frac{r\,\gamma}{1+\gamma} (-3.6 \times 10^{-6}) \Bigg] .\end{aligned}$$

Which, upon further substitution gives

$$\begin{aligned}\Delta b = (1+.26) \Bigg[& \frac{1.25(1 - 2\times.26)}{28.8 \times 10^6} \left(\frac{(1.25)^2 \times 153 - (2.44)^2 \times 255}{(2.44)^2 - (1.25)^2}\right) + \\ & \frac{(1.25)^2\,(2.44)^2}{1.25\left((2.44)^2 - (1.25)^2\right)} \left(\frac{153-255}{28.8 \times 10^6}\right) - \frac{1.25 \times .26}{1+.26}(-3.6 \times 10^{-6}) \Bigg]\end{aligned}$$

$$\Delta b = -14 \times 10^{-6} \text{ in.}$$

which is the change in b due to pressure.

To calculate Δa for the shaft it is assumed that the shaft is at a uniform temperature throughout. Therefore

$$\Delta a = a\alpha_1 (t - 68)$$

$$= 1.25 \times 5.9 \times 10^{-6} (92 - 68) = 177 \times 10^{-6} \text{ in.}$$

The effect of centrifugal force on the shaft radius is negligible for all runs of the experimental data. Assuming the worst possible case, i.e., the shaft to be a thin disc rotated at 12,000 rpm, then*

$$\Delta a = (1 - \gamma^2)\,\frac{\rho\,\omega^2 a^3}{gE}\left[\frac{2}{8(1+\gamma)}\right]$$

$$= \left(1 - (.26)^2\right)(.29)\,\frac{\left(\frac{12{,}000 \times 2\pi}{60}\right)^2 (1.25)^3}{32.2 \times 12 \times 28.8 \times 10^6}\left[\frac{2}{8(1+.26)}\right]$$

$$= 14.9 \times 10^{-6}\text{in.}$$

The Δa calculated above due to centrifugal force assumes the worst possible condition used in this experiment and is relatively small in comparison to the tolerance for reference measurements; therefore, the effect of centrifugal force on the change of clearance will be neglected.

The total change in radial clearance is the sum of the changes of Δb due to temperature and pressure, and Δa, i.e.

$$\Delta h = \left[-14 + 176 \quad -177\right]10^{-6} \text{ in.}$$
$$= -13 \times 10^{-6} \text{ in.}$$

The diametral clearance at 68°F. is

$$C_{\text{reference}} = 2.5041 - 2.5003 = 3800 \times 10^{-6} \text{ in.}$$

The actual clearance is therefore

$$C_{\text{actual}} = \left[3800 - 2 \times 13\right] 10^{-6} = 3774 \times 10^{-6} \text{ in.}$$
$$C^3 = 5.37 \times 10^{-8} \text{ in.}^3$$

---

* See reference (20).

The measured mass rate of flow through bearing 1 is

$$M_1 = 25 \text{ lbs.}/2 \text{ min.}-15.8 \text{ sec.} = 11.01 \text{ lb./min.}$$

Correspondingly, the mass rate through bearing 4 is

$$M_4 = \frac{25 \text{ lbs.}}{1 \text{ min.}-51.2 \text{ sec.}} = 13.48 \text{ lb./min.}$$

The velocities through bearings 1 and 4 are*

$$U_1 = \frac{11.01 \times .01611 \times 144 \times 2}{60 \times 2.5 \pi \times 3772 \times 10^{-6}} = 28.7 \text{ ft./sec.}$$

and

$$U_4 = \frac{13.48 \times .01611 \times 144 \times 2}{60 \times 2.5 \pi \times 3772 \times 10^{-6}} = 35.2 \text{ ft./sec.}$$

The Reynolds Numbers through bearings 1 and 4 are **

$$R = \frac{mU}{\nu} = \frac{CU}{4\nu}$$

$$R_1 = \frac{3772 \times 10^{-6} \times 28.7}{4 \times 12 \times 8.1 \times 10^{-6}} = 278$$

and

$$R_2 = \frac{3772 \times 10^{-6} \times 35.2}{4 \times 12 \times 8.1 \times 10^{-6}} = 341 .$$

The rotational flow dimensionless parameter is

$$\frac{rm\omega}{\nu} = 0, \text{ since } \omega = 0.$$

The third dimensionless parameter is

$$\frac{m^3 P}{\nu^2 \rho L} = \frac{5.37 \times 10^{-8} \times .01611 \times 204 \times 32.2}{64(8.1 \times 10^{-6})^2 \times 2} = 676 .$$

* See reference (11) for specific volume.
** See figure (6) for kinematic viscosity.

PLOT OF KINEMATIC VISCOSITY OF WATER VERSUS TEMPERATURE

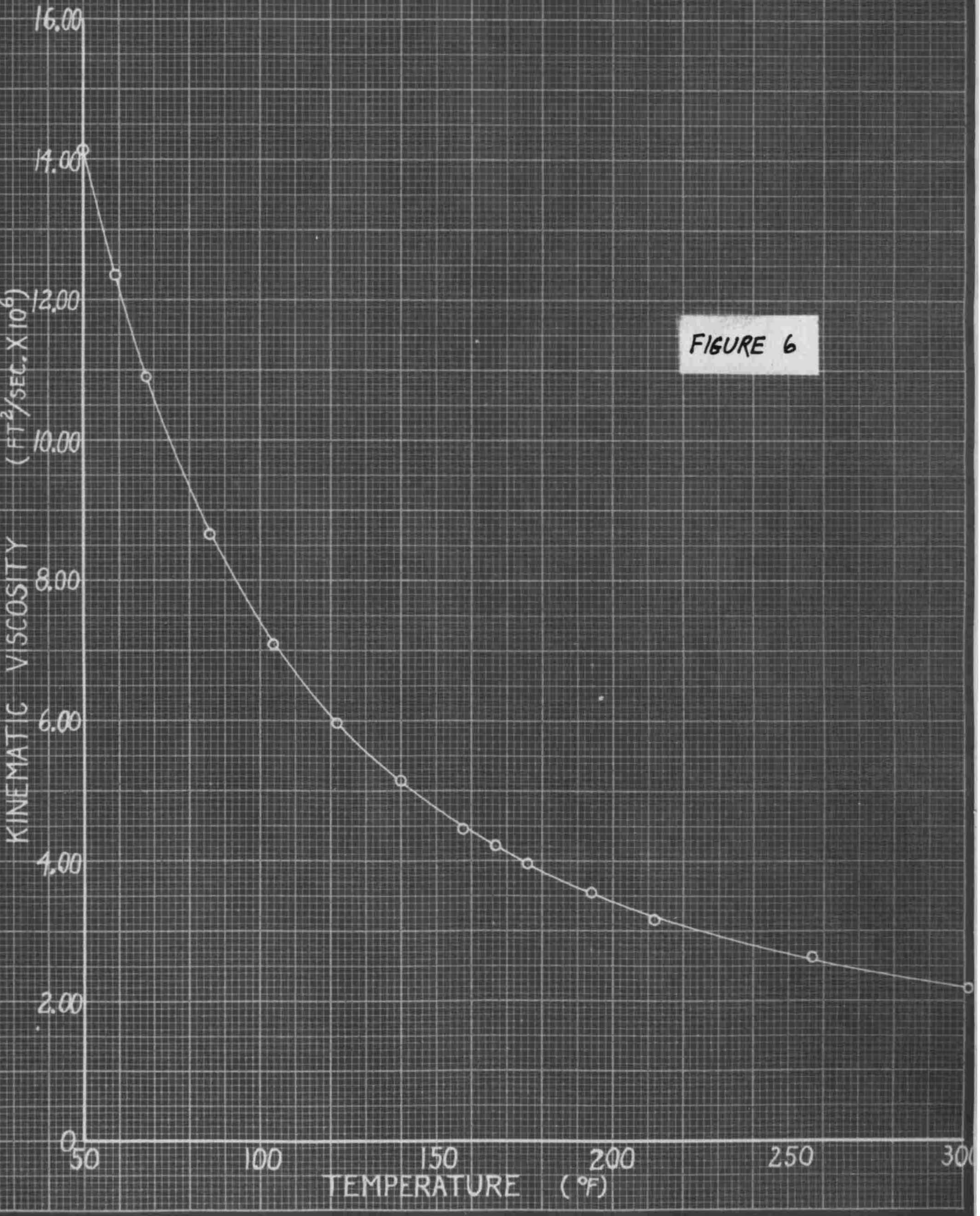


FIGURE 6

The other points for plotting the curves shown in figure 11 were obtained in a similar manner. It may be noted that no consideration of temperature gradient across the bearing was made in the above computations. Temperature measurements, shown in Data Sheet and Calculated Data, indicated this gradient was negligably small for all the test runs made in this investigation. Further, it was found that the change of temperature of the water flowing through the bearing was small, being a maximum of 4°F. for all runs except no. 58. The indicated rise of 8°F. for this run appeared to be erroneous. With these small temperature rises, the authors felt justified in using the arithmetic mean temperature of the water flowing through the bearing for determining the viscosities used in their computations.

The authors were unable to obtain equal flow from bearings 1 and 4. The apparent trouble was that bearing 4 was slightly skewed or restrained in the inner housing. This may best be shown by figure 7 and the forthcoming explanation.

Figure 7 is a plot of dimensionless parameters for $\frac{rm\omega}{\nu}$ equal to 0 and 440 for bearings 1 and 4. For comparison, a plot of Cornish's curve for $\frac{rm\omega}{\nu}$ equal 0 is shown by a dashed line. The lower or straight portion of this line is the theoretical plot for laminar flow as given by equation 4. As the majority of the runs for $\frac{rm\omega}{\nu} = 440$ were made at approximately 4000 RPM, we find

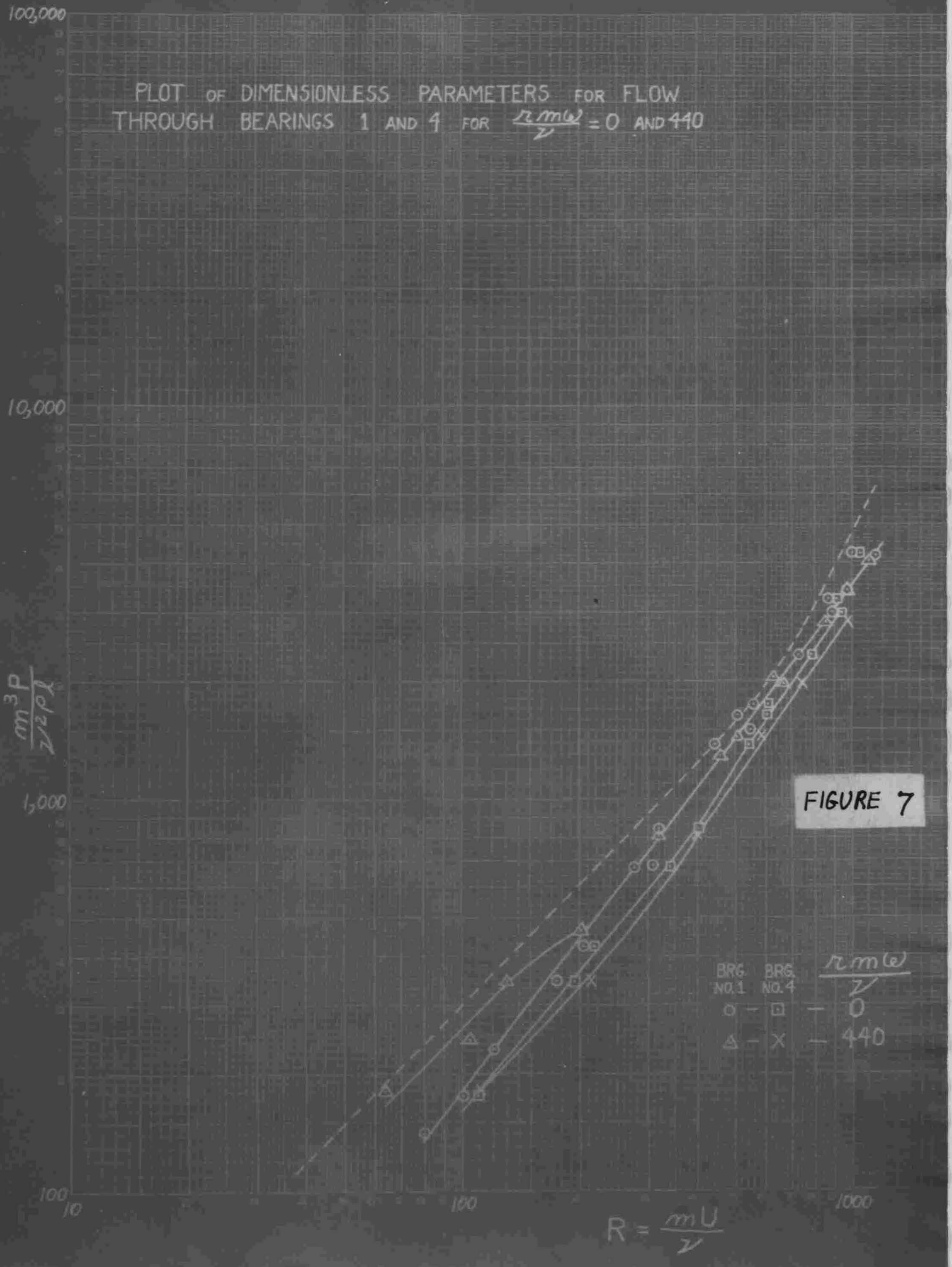

PLOT OF DIMENSIONLESS PARAMETERS FOR FLOW THROUGH BEARINGS 1 AND 4 FOR $\frac{rm\omega}{\nu} = 0$ AND 440

FIGURE 7

$$\frac{2\omega am}{\nu}\left(\frac{a+b}{m}\right)^{-\frac{1}{2}} \approx 17.2$$

for this speed. Referring to figure 4, we find this point well below the curve and thus the effect of this speed of rotation on turbulence may be neglected. Therefore, *if the bearings and journal are concentric*, we can expect the curves for $\frac{rm\omega}{\nu} = 0$ and $\frac{rm\omega}{\nu} = 440$ to coincide.

The curves for bearing 1 do not coincide at the lower pressures due to the 100 lb. load on the middle bearings tending to cause eccentricity and resulting in an increase of flow when the rotor is stationary; while at 4000 RPM the effect of this load on eccentricity is negligibly small.* As the pressure drop and resulting flow is increased the eccentricity decreases as is to be expected, and the curves for bearing 1 coincide.

An inspection of figure 7 shows that the curves for bearing 4 are very close at the lower Reynolds Numbers but do not coincide at any point. Furthermore, the curve for $\frac{rm\omega}{\nu} = 440$ is slightly lower than the curve for $\frac{rm\omega}{\nu} = 0$, indicating an increase of flow with rotation. Such results are not normal for an unrestrained bearing as the eccentricity should decrease with rotation resulting in a decrease of flow as shown by equation (5) in Chapter III. The increase of flow with rotation can then be explained by the pumping action of an eccentric journal as shown

*A full discussion of eccentricity follows shortly.

by equation (7). Therefore, bearing 4 must be restrained or skewed in its housing.

Although extreme care was exercised in the manufacture and assembly of these bearings it is virtually impossible to insure concentricity of upper and lower bearings in this type of apparatus unless the bearings are concentrically bored in place. Due to the time required, the faults in bearing 4 were not corrected. It was decided to include only the results obtained from bearing 1 in the final results of this investigation. Figure 11 is a dimensionless plot of the results obtained from bearing 1.

Let us now consider the effects of eccentricity on our problem. By equation (5), we can expect up to 150% increase in rate of flow for laminar flow with an eccentric rotor. In the turbulent flow region the effect of eccentricity is much less, Cornish finding the increase not more than 30% for maximum eccentricity.

To estimate the eccentricity of the journal we may refer to Kingsbury's curve of leakage factors and to Sommerfeld's chart for minimum film thickness. Sommerfeld's chart gives the minimum film thickness for a bearing without end leakage. Our problem is to obtain the minimum film thickness or clearance with axial flow or end leakage. We may do this by applying a correction in the form of a leakage factor obtained from Kingsbury's curve. The above curves are reproduced in figures 8 & 9 respectively.*

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* Figures 8 and 9 are from reference (13)

LEAKAGE FACTOR FOR LOAD FOR BEARINGS
OF MINIMUM COEFFICIENT OF FRICTION

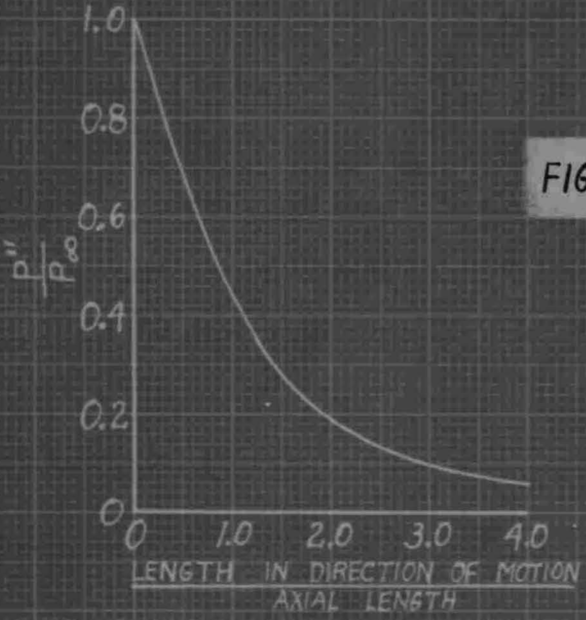


FIGURE 8

SOMMERFELD CHART FOR FULL JOURNAL BEARING
WITH UNIFORM VISCOSITY WITHOUT SIDE LEAKAGE

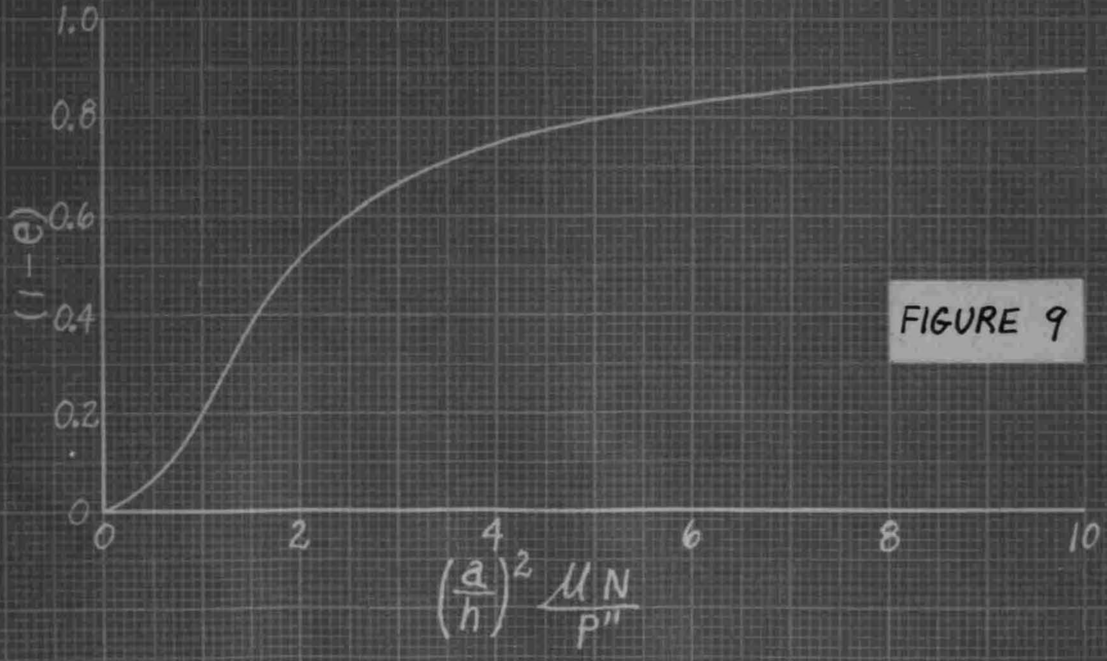


FIGURE 9

We must first obtain the leakage factor from figure 8. Following Kingsbury's practice, our bearing may be assumed effectively that of a partial bearing of 120° arc. The length in the direction of motion then equals $\pi D/3$ or 2.62in. The abscissa for entry in figure 8 then becomes 2.62/2, or 1.31. From figure 8 we find $\frac{P}{P_{\infty}}$ is about 0.33. This means that with axial flow the bearing will carry only about 33% of the load that would give the same minimum clearance in a bearing without axial flow, i.e., end leakage causes the journal to ride lower in the bearing. As the load on each individual bearing is 50 lb, P"is $\frac{50}{2.5X2}$, or 10 psi. P_{∞} is then equal to 10/.33, or 30.3 psi. The apparent bearing load, P_{∞} , is used instead of P" for computing the abscissa for entry in figure 9.

We can now estimate the eccentricity by use of figure 9. For run no. 29 we have given:

$r = 1.25$ in.

$h = 1.884 X 10^{-3}$ in.

$t = 95^{\circ}F.$ and thus $\mu = 10.5^{6} X 10^{-8}$ lb-sec/in^2 *

$P'' = 30.3$ psi

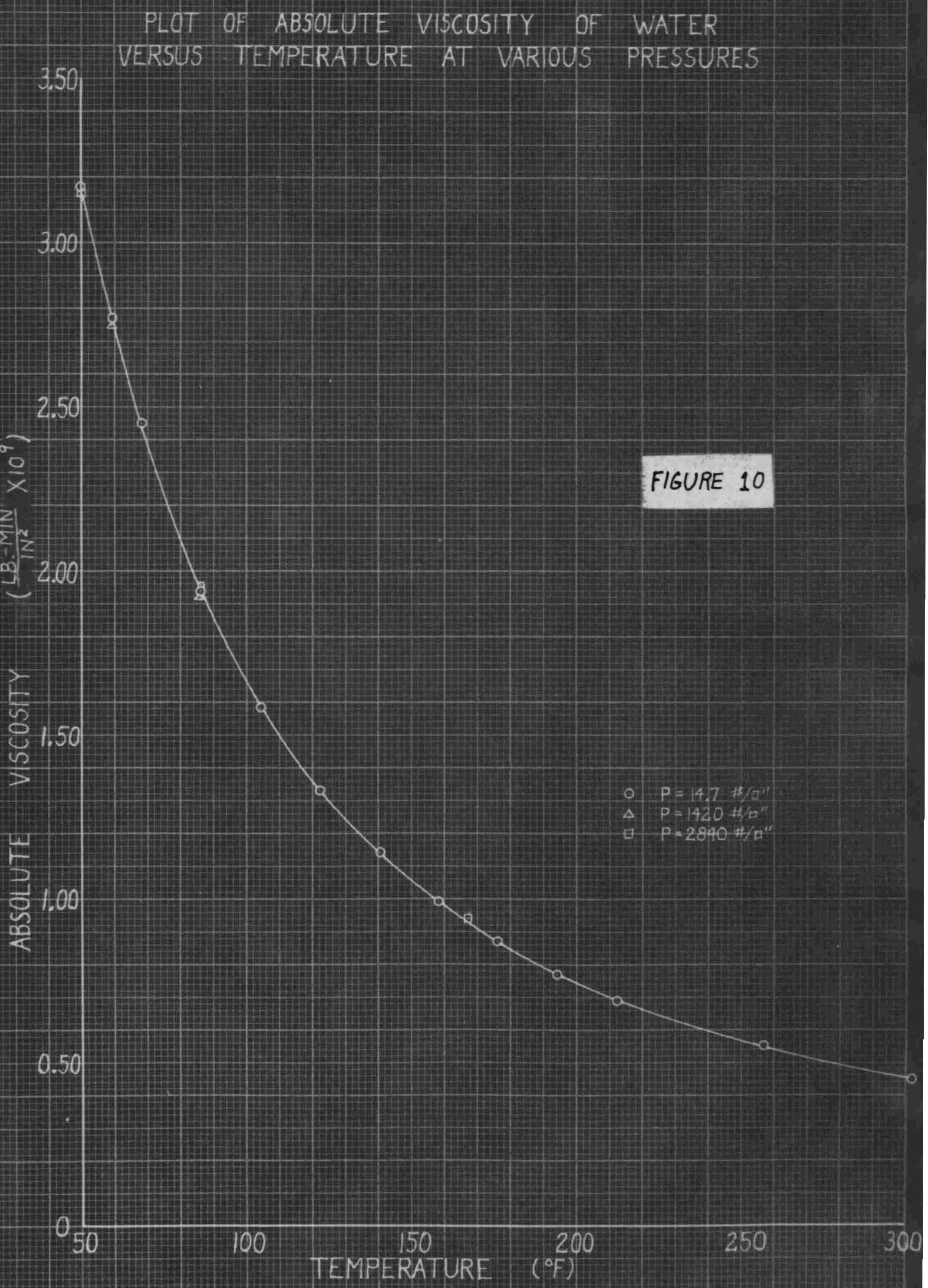
$N = 4000$ RPM

then

$$(r/h)^2 \; \frac{\mu N}{P''} = \left(\frac{1.25}{0.001884}\right)^2 X \; \frac{10.5_6 X 10^{-8} X 4000}{30.3}$$

$$= 6.13$$

*See Fig.10 for absolute viscosity.



PLOT OF ABSOLUTE VISCOSITY OF WATER VERSUS TEMPERATURE AT VARIOUS PRESSURES

FIGURE 10

From figure 9, we find

$$1 - e = 0.84$$

or

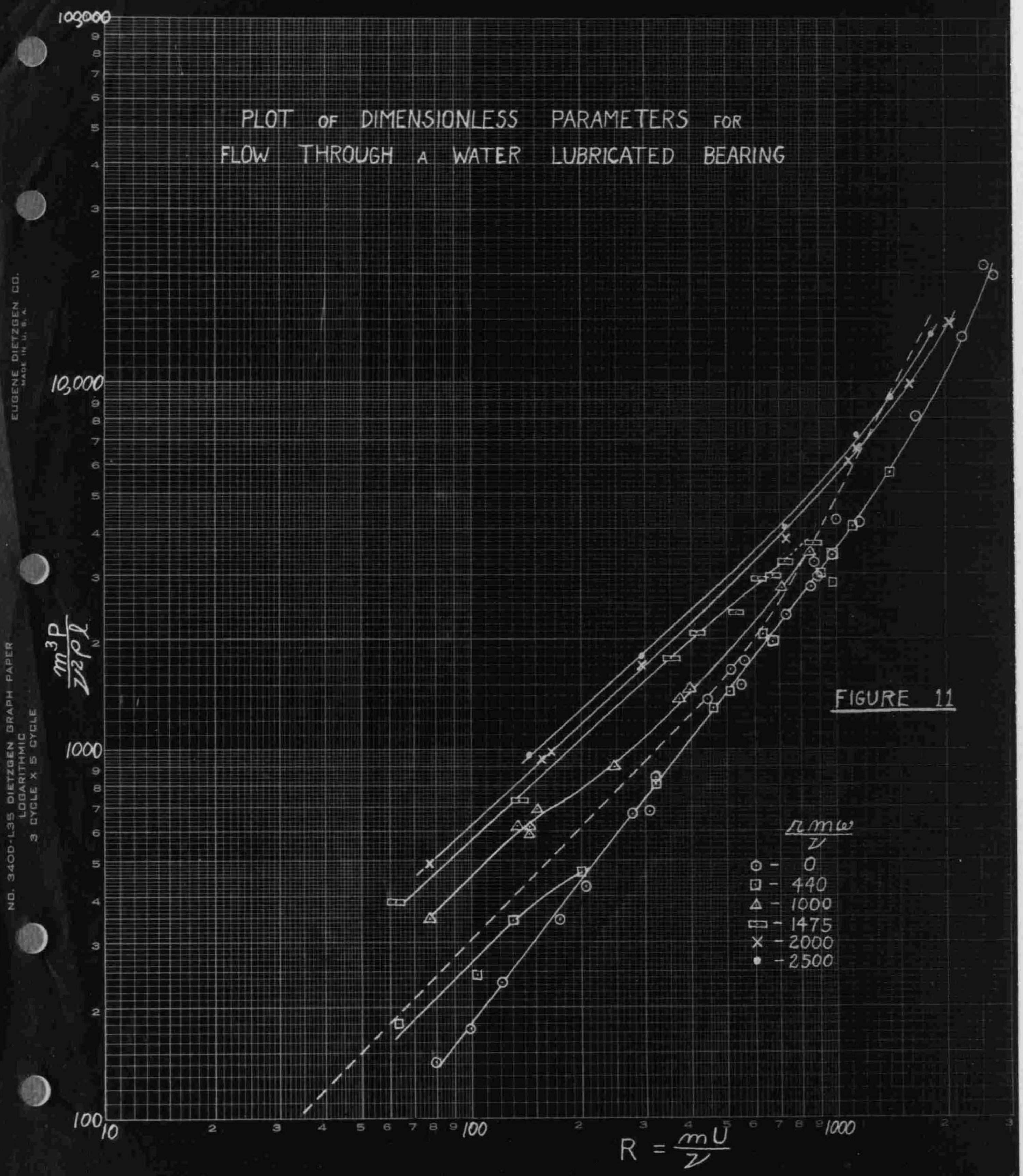
$$e = 0.16$$

Then, the increase in flow is given by 1.5 e^2 or about 3.84 percent. Considering the accuracies possible in the determination of some of the other factors in this experiment this small percentage is negligible. The eccentricity decreases fairly rapidly with increase of speed.

Brief mention should be made of the effect of gravity. The static head for bearing 1 was approximately one foot, or .433 psi. This figure is negligibly small in comparison with the pressure drops encountered in this investigation.

It may be observed that the laminar flow line obtained by the authors as shown in figure 11 lies somewhat below the theoretical laminar flow line shown in dashed lines in the same figure. While, at the first glance, the discrepancy appears rather large, actually the error is well within the accuracy of measurements. To illustrate, suppose we apply our limits of accuracy of measurements to a specified point on our laminar flow line and see if it will raise the point up to the theoretical laminar flow line. For run 29 we have given:

Calculated clearance, C	3768×10^{-6} in.
Limits of accuracy of meas.	$\pm 200 \times 10^{-6}$ in.
Maximum possible clearance, C_{max}.	4168×10^{-6} in.


PLOT OF DIMENSIONLESS PARAMETERS FOR FLOW THROUGH A WATER LUBRICATED BEARING
FIGURE 11
100,000
10,000
1000
100
10
100
1000
$\frac{m^3 P}{\nu^2 \rho l}$
$R = \frac{mU}{\nu}$
$\frac{rm\omega}{\nu}$
○ - 0
⊡ - 440
△ - 1000
▭ - 1475
× - 2000
● - 2500
EUGENE DIETZGEN CO. MADE IN U. S. A.
NO. 340D-L35 DIETZGEN GRAPH PAPER LOGARITHMIC 3 CYCLE X 5 CYCLE

$$C^3 \text{ max.} = 7.24 \times 10^{-8} \text{ in.}^3$$

The mean velocity through the bearing then becomes

$$U = \frac{12.2 \times 0.01611 \times 144 \times 2}{60 \times 2.5 \times \pi \times 4168 \times 10^{-6}} = 28.8 \text{ ft./sec.}$$

The Reynolds Number, R, is then

$$R = \frac{m\,U}{\nu} = \frac{4168 \times 10^{-6} \times 28.8}{4 \times 12 \times 7.85 \times 10^{-6}} = 319$$

as before.

The pressure parameter is then

$$\frac{m^3\,P}{\nu^2 \rho L} = \frac{7.24 \times 10^{-8} \times 0.01611 \times 230 \times 32.2}{64 \times 61.6 \times 10^{-6} \times 2} = 1096$$

versus the previous value of 810. Applying this value of $\frac{m^3\ P}{\nu^2 \rho L}$ brings this point on our curve for $\frac{rm\omega}{\nu} = 440$ well above the theoretical laminar flow line.

Recognizing the fact that our curve as shown in figure 11 is slightly below the theoretical plot, the authors feel that the error is sufficiently accounted for by the limits of accuracy of the reference measurements. Due consideration of this deviation from the theoretical laminar flow line is necessary when figure 11 is used for obtaining the flow in similar bearings.

BIBLIOGRAPHY

1. American Society for Metals. Metals handbook. Cleveland, ASM, 1948.

2. ASTM and ASME. Symposium on effect of temperature on the properties of metals. Chicago, ASTM and ASME, 1931.

3. Cornish, R. J. Flow of water through fine clearances. Proceedings of the royal society of London. A,140: 227-240, 1933.

4. Davies, S. J. and White, C. M. An experimental study of the flow of water in pipes of rectangular section. Proceedings of the royal society of London. A,119:92-106, 1928.

5. Dorsey, N. E. Properties of ordinary water-substance. New York, Reinhold, 1940.

6. Exline, Paul G. Leakage in capilliary seals of hydraulic valves and pumps. Product engineering. 290-296, April 1946.

7. Fage, A. The influence of wall oscillations, wall rotation, and entry eddies, on the breakdown of laminar flow in an annular pipe. Proceedings of the royal society of London. A,165: 501-529, 1938.

8. Goldstein, S. Stability of viscous fluid flow between rotating concentric cylinders. Proceedings of the Cambridge philosophical society. 33: 41-61, 1937.

9. Goldstein, S. Modern developments in fluid dynamics. Vols. 1 and 2. London, Oxford, 1938.

10. Hersey, Mayo Dyer. Theory of lubrication. New York, John Wiley and Sons, 1936.

11. Keenan, Joseph H. and Keyes, Frederick G. Thermodynamic properties of steam. New York, John Wiley and Sons, 1936.

12. Lamb, Sir Horace. Hydrodynamics. New York, Dover, 1945.

13. Norton, Arthur Edwin. Lubrication, New York, McGraw-Hill, 1942.

14. Orloff, P. I. Coefficient of friction, oil flow and

heat balance of a full journal bearing. Aeronautical engineering (Moscow). 25-56, 1935. (National advisory committee for aeronautics technical memorandum no. 1165; October 1947).

15. Stodola, A. Steam and gas turbines. Vol. 1. New York, McGraw-Hill, 1927.

16. Suzuki, S. On the leakage of water through clearance space. Journal of faculty of engineering, Tokyo Imperial University, July 1929.

17. Taylor, G. I. Fluid friction between rotating cylinders. Proceedings of the royal society of London. A,157; 546-578, 1936.

18. Taylor, G. I. On stability of a viscous fluid contained between two rotating cylinders. Philosophical transactions of the royal society. A,223: 289-344, 1923.

19. Thompson, F. C. Kenneford, A. S., and Seager, G. C. The tensile stresses in a bearing metal cast on to a strip and the fatigue failure of bearings. Engineering. 235 +, Aug. 26, 1938.

20. Timoshenko, S. Strength of materials, parts 1 and 2. Lancaster, D. Van Nostrand, 1940 and 1941.

APPENDIX I

DERIVATION OF FORMULA FOR LAMINAR FLOW BETWEEN CONCENTRIC CYLINDERS

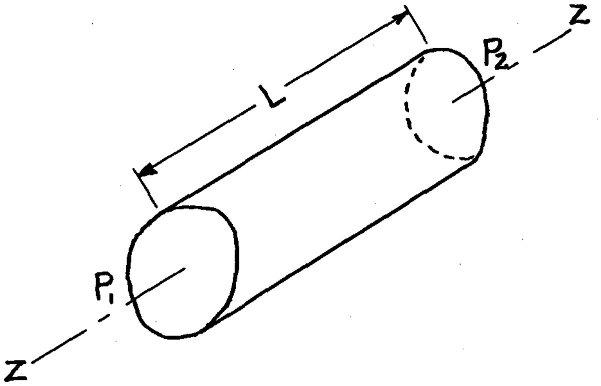


Figure 12

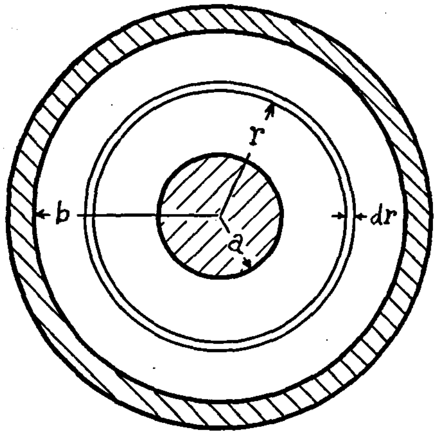


Figure 13

Referring to figures 12 and 13 above, assume steady flow of the liquid straight through the annular uniform cross-sectional area. Taking the Z axis coincident with the axis of the tube, and assuming the velocity is parallel to the Z axis and a function of the radial distance, r, from this axis, the shear stress across a plane perpendicular to r will be:

shear stress = abs. viscosity x velocity gradient, or in differential notation

$$\tau = \mu \frac{\partial U}{\partial r} .$$

Hence, considering a cylindrical shell of fluid whose bounding radii are r and $(r + dr)$ the differential retarding tangential force on the two curved surfaces is

$$dF = - \frac{\partial}{\partial r} \left[\mu \frac{\partial U}{\partial r} \cdot 2\pi r L \right] dr \quad . \qquad (16)$$

Since steady flow was assumed, this force must be balanced by normal pressures on the plane ends of the shell.

The velocity U is independent of the distance along the flow path, i.e.,

$$\frac{\partial U}{\partial Z} = 0$$

The differential force on the cylindrical shell due to the normal pressures is

$$dF = \left(p_1 - p_2\right) 2\pi r \; dr \quad . \tag{17}$$

Equating the two differential forces, we have

$$- \frac{\partial}{\partial r} \left[\mu \frac{\partial U}{\partial r} \; 2\pi rL\right] dr = (p_1 - p_2) 2\pi r \; dr$$

hence

$$\frac{\partial}{\partial r} \left(\frac{r \, \partial U}{\partial r}\right) = - \frac{(p_1 - p_2) \; r}{\mu L} \quad . \tag{18}$$

The integral of equation (18) is

$$U = - \frac{(p_1 - p_2)}{4 \mu L} r^2 + A \log r + B. \tag{19}$$

Assuming there is no slippage of the fluid at the outer surface of the shaft or inner surface of the outer shell, the boundary conditions are:

$U = 0$ at $r = a$

and $U = 0$ at $r = b$.

Substituting boundary conditions in (19), we have

$$0 = -\frac{p_1 - p_2}{4\mu L} a^2 + A \ln a + B \qquad \text{(a)}$$

and

$$0 = -\frac{p_1 - p_2}{4\mu L} b^2 + A \ln b + B. \qquad \text{(b)}$$

Then, by subtracting (b) from (a), we have

$$A (\ln a - \ln b) = \frac{p_1 - p_2}{4\mu L} (a^2 - b^2)$$

or

$$A = \frac{p_1 - p_2}{4\mu L} \frac{(a^2 - b^2)}{\ln\left(\frac{a}{b}\right)} .$$

Substituting the above value of A in equation (a), we obtain

$$B = \frac{p_1 - p_2}{4\mu L} a^2 - \frac{p_1 - p_2}{4\mu L} \frac{(a^2 - b^2)}{\ln\left(\frac{a}{b}\right)} \ln a.$$

Substituting values of the constants A and B in equation (19), we have

$$U = -\frac{p_1 - p_2}{4\mu L} r^2 + \frac{p_1 - p_2}{4\mu L} \left[\frac{a^2 - b^2}{\ln\left(\frac{a}{b}\right)}\right] \ln r + \frac{p_1 - p_2}{4\mu L} a^2 - \left[\frac{p_1 - p_2}{4\mu L} \frac{(a^2 - b^2)}{\ln\left(\frac{a}{b}\right)}\right] \ln a$$

which upon simplification yields

$$U = \frac{p_1 - p_2}{4\mu L} \left[(a^2 - r^2) + \frac{a^2 - b^2}{\ln\left(\frac{a}{b}\right)} \ln\left(\frac{r}{a}\right)\right] . \qquad (20)$$

Equation (20) is an expression for the axial velocity at any radius r.

The volume rate of flow, Q, is therefore

$$Q = \int_a^b U \, dA = \int_a^b U \, 2\pi r \, dr$$

which upon substitution for U becomes

$$Q = \int_a^b \frac{p_1-p_2}{4\mu L}\left[(a^2-r^2) + \frac{a^2-b^2}{\ln\left(\frac{a}{b}\right)}\left(\ln \frac{r}{a}\right)\right] 2\pi r \, dr$$

and upon integration yields

$$Q = \frac{\pi(p_1-p_2)}{8\mu L}\left[(b^4-a^4) - \frac{(b^2-a^2)^2}{\ln\left(\frac{b}{a}\right)}\right] . \qquad (21)$$

This formula is difficult to use as the numerical values must be carried out 12 to 15 significant figures for accurate results. We may simplify equation (21) by substituting for b its equivalent, a + h. Then

$$Q = \frac{\pi(p_1-p_2)}{8\mu L}\left[\left((a+h)^4 - a^4\right) - \frac{\left((a+h)^2 - a^2\right)^2}{\ln\left(\frac{a+h}{a}\right)}\right]$$

$$= \frac{\pi(p_1-p_2)}{8\mu L}\left[(a+h)^2 - a^2\right]\left[(a+h)^2 + a^2 - \frac{(a+h)^2 - a^2}{\ln\left(1+\frac{h}{a}\right)}\right] .$$

Since

$$\ln\left(1+\frac{h}{a}\right) = \frac{h}{a} - \frac{1}{2}\left(\frac{h}{a}\right)^2 + \frac{1}{3}\left(\frac{h}{a}\right)^3 - \frac{1}{4}\left(\frac{h}{a}\right)^4 + \cdots \text{ where } -1 < \frac{h}{a} < 1$$

Then

$$Q = \frac{\pi(p_1-p_2)}{8\mu L}\left[a^2+2ah+h^2-a^2\right]\left[2a^2+2ah+h^2 - \frac{(2ah+h^2)}{\frac{h}{a}-\frac{1}{2}\left(\frac{h}{a}\right)^2+\frac{1}{3}\left(\frac{h}{a}\right)^3-\frac{1}{4}\left(\frac{h}{a}\right)^4+\cdots}\right]$$

$$Q = \frac{\pi(p_1-p_2)}{\mu 8 L}\left[h^2+2ah\right]\left[(h^2+2ah+2a^2)-a^2\left(2+2\frac{h}{a}+\frac{1}{3}\left(\frac{h}{a}\right)^2-\frac{1}{90}\left(\frac{h}{a}\right)^4+\cdots\right)\right]$$

$$= \frac{\pi(p_1-p_2)}{8\mu L}\, ah\left[\frac{h}{a}+2\right]\left[h^2\left(1+\frac{2a}{h}+\frac{2a^2}{h^2}\right) - h^2\left(\frac{2a^2}{h^2}+\frac{2a}{h}+\frac{1}{3}-\frac{1}{90}\left(\frac{h}{a}\right)^2+\cdots\right)\right]$$

$$= \frac{\pi(p_1-p_2)\quad ah^3}{6\mu L}\left[\frac{6}{8}\;\frac{h}{a}+\frac{3}{2}\right]\left[\frac{2}{3}+\frac{1}{90}\left(\frac{h}{a}\right)^2+\cdots\right]$$

and finally

$$Q = \frac{\pi(p_1-p_2)\quad ah^3}{6\mu L}\left[1+\frac{1}{2}\;\frac{h}{a}+\frac{1}{60}\left(\frac{h}{a}\right)^2+\frac{1}{120}\left(\frac{h}{a}\right)^3+\cdots\right].$$

Where $\frac{h}{a}$ is very small number the above equation reduces to:

$$Q = \frac{\pi(p_1-p_2)ah^3}{6\mu L}\,, \tag{22}$$

APPENDIX II

DERIVATION OF FORMULA FOR CHANGE OF CLEARANCE WITH CHANGE OF TEMPERATURE AND PRESSURE

In the following derivation the wall of the bearing will be considered that of a cylinder which is non-uniformly heated thus setting up thermal stresses. It is assumed that the bearing is a homogeneous material, or if a lining is used it is so thin that its effect can be neglected. The effect of pressure on the inner and outer surfaces, and on the ends of the bearing are also taken into account. The temperature distribution is assumed symmetrical with respect to the axis and constant along the axis. Therefore the deformation is symmetrical about the axis.

A ring is cut from the cylinder by two cross-sections perpendicular to the axis and unit distance apart. It is assumed that the cylinder is of sufficient length so that the ends of the element remain plane.

The Z axis is taken as the axis of the cylinder and W is the displacement in the Z direction. The subscripts r and t denote the radial and circumferential directions respectively. The symbol V denotes displacement in the radial direction.

The unit deformations in the three perpendicular directions of an element in the above ring are:

$$\begin{aligned} \epsilon_z &= \frac{dW}{dZ} = \text{constant} \\ \epsilon_r &= \frac{dV}{dr} \qquad\qquad (a) \\ \epsilon_t &= \frac{V}{r} \quad . \end{aligned}$$

The strain can be represented as functions of the stresses σ_z, σ_r, σ_t, and the temperature. Then

$$\begin{aligned} \epsilon_z &= \frac{\sigma_z}{E} - \frac{\gamma}{E}\left(\sigma_r + \sigma_t\right) + \alpha T \\ \epsilon_r &= \frac{\sigma_r}{E} - \frac{\gamma}{E}\left(\sigma_z + \sigma_t\right) + \alpha T \qquad\qquad (b) \\ \epsilon_t &= \frac{\sigma_t}{E} - \frac{\gamma}{E}\left(\sigma_z + \sigma_r\right) + \alpha T \end{aligned}$$

Where α is the linear coefficient of thermal expansion and T the temperature difference through the bearing which varies with the radial distance, r.

The unit increase in volume is

$$\delta = \epsilon_z + \epsilon_r + \epsilon_t = \frac{1-2\gamma}{E}\left(\sigma_z + \sigma_r + \sigma_t\right) + 3\alpha T.$$

From the foregoing equations, the stresses are

$$\begin{aligned} \sigma_z &= \frac{E}{1+\gamma}\left(\epsilon_z + \frac{\gamma}{1-2\gamma}\delta\right) - \frac{\alpha TE}{1-2\gamma} \\ \sigma_r &= \frac{E}{1+\gamma}\left(\epsilon_r + \frac{\gamma}{1-2\gamma}\delta\right) - \frac{\alpha TE}{1-2\gamma} \qquad\qquad (c) \\ \sigma_t &= \frac{E}{1+\gamma}\left(\epsilon_t + \frac{\gamma}{1-2\gamma}\delta\right) - \frac{\alpha TE}{1-2\gamma} \end{aligned}$$

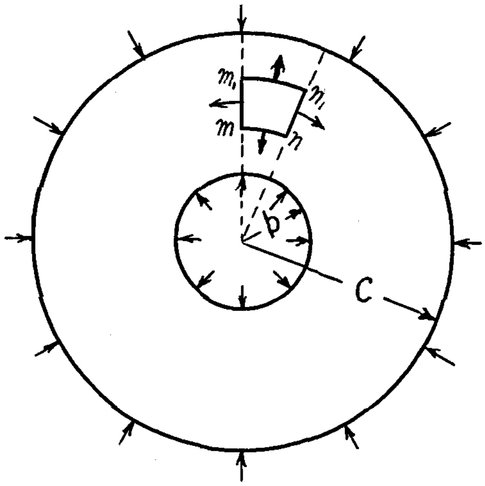


Figure 14

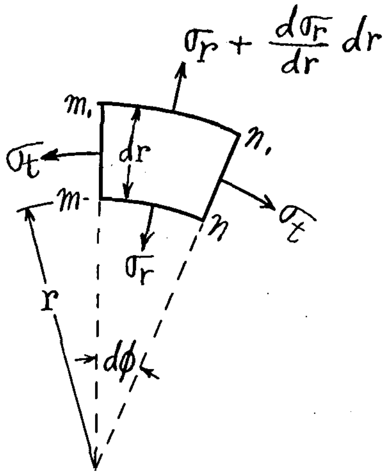


Figure 15

Referring to figures 14 and 15 above:

σ_t = normal hoop stress acting on the sides mm_1, and nn_1 of the element.

σ_r = normal radial stress on the side mn. This stress varies with the radius r and changes by an amount $\left(\frac{d\sigma_r}{dr}\right)dr$ in the distance dr.

The normal radial stress on the side m_1n_1 is

$$\sigma_r + \frac{d\sigma_r}{dr}\,dr.$$

Summing up the forces on the element in the direction of the bisector of the angle $d\phi$ yields the following equation of equilibrium, neglecting the mass of the element.

$$\sigma_r\, r\,d\phi + \sigma_t\, dr\, d\phi - \left(\sigma_r + \frac{d\sigma_r}{dr}\,dr\right)(r + dr)\,d\phi = 0.$$

Neglecting small quantities

$$\sigma_t - \sigma_r - r \frac{d\sigma_r}{dr} = 0.$$

Substituting the stress and unit deformation equations in the above equation of equilibrium yields

$$\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} - \frac{V}{r^2} = \frac{1+\gamma}{1-\gamma}\alpha\frac{dT}{dr} \quad .$$

This differential equation determines the radial displacement V for any particular distribution of temperature. This expression may be written in the form

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rV)\right] = \frac{1+\gamma}{1-\gamma}\alpha\frac{dT}{dr} .$$

Integrating with respect to r, we have

$$\frac{d}{dr}(rV) = \frac{1+\gamma}{1-\gamma}\alpha\,Tr + C_1' r \quad .$$

Integrating again, yields

$$V = \frac{1}{r}\frac{1+\gamma}{1-\gamma}\int_b^r \alpha\,Tr\,dr + C_1 r + C_2\frac{1}{r} \quad . \qquad (23)$$

The above equation contains two constants C_1 and C_2 which must be evaluated from the boundary conditions. Calling the outside pressure on the bearing P_o and the inside pressure P_i, then

$$(\sigma_r)_{r=a} = -P_i \quad \text{and} \quad (\sigma_r)_{r=b} = -P_o \quad .$$

The negative sign is used because normal stress is taken as negative for compression. A general expression for σ_r is obtained by substituting $\epsilon_r = \frac{dV}{dr}$, $\epsilon_t = \frac{V}{r}$ and the foregoing expression for V into the stress-strain equation for σ_r given in equation (c). These substitutions give

$$\sigma_r = \frac{E}{1+\gamma}\left[-\frac{1+\gamma}{1-\gamma}\,\frac{1}{r^2}\int_b^r \alpha\, Tr\, dr + \frac{C_1}{1-2\gamma} - \frac{C_2}{r^2} + \frac{\gamma}{1-2\gamma}\epsilon_z\right]. \quad (24)$$

To evaluate constants C_1 and C_2 the boundary conditions are substituted in the above general stress equation. The boundary conditions are:

$$(\sigma_r)_{r=b} = -P_i$$
$$(\sigma_r)_{r=c} = -P_o \ .$$

Substituting, we get

$$-p_i = \frac{E}{1+\gamma}\left[-\frac{1+\gamma}{1-\gamma}\,\frac{1}{b^2}\int_b^b \alpha\, Tr\, dr + \frac{C_1}{1-2\gamma} - \frac{C_2}{b^2} + \frac{\gamma}{1-2\gamma}\epsilon_z\right] \quad (25)$$

and

$$-p_o = \frac{E}{1+\gamma}\left[-\frac{1+\gamma}{1-\gamma}\,\frac{1}{c^2}\int_b^c \alpha\, Tr\, dr + \frac{C_1}{1-2\gamma} - \frac{C_2}{c^2} + \frac{\gamma}{1-2\gamma}\epsilon_z\right]. \quad (26)$$

Subtracting, we have

$$\frac{(p_i - p_o)(1+\gamma)}{E} = -\frac{1+\gamma}{1-\gamma}\,\frac{1}{c^2}\int_b^c \alpha\, Tr\, dr + \frac{C_2}{b^2} - \frac{C_2}{c^2}$$

or

$$C_2 = \frac{b^2\ c^2}{c^2 - b^2} \left[\frac{(pi-po)(1+\gamma)}{E} + \frac{1+\gamma}{1-\gamma}\ \frac{1}{c^2} \int_b^c \propto Tr\ dr \right]. (27)$$

Substituting the above expression for C_2 in formula (25), we have

$$-pi = \frac{E}{1+\gamma} \left[\frac{C_1}{1-2\gamma} - \frac{c^2}{c^2 b^2} \left(\frac{(pi-po)(1+\gamma)}{E} + \frac{1+\gamma}{1-\gamma}\ \frac{1}{c^2} \int_b^c \propto Tr\ dr \right) + \frac{\gamma}{1-2\gamma} \epsilon_Z \right]$$

or,

$$C_1 = \frac{(1+\gamma)(1-2\gamma)}{E} \left(\frac{b^2 pi - c^2\ po}{c^2 - b^2} \right) + \left(\frac{1+\gamma}{1-\gamma} \right) \frac{1-2\gamma}{c^2 b^2} \int_b^c \propto Trdr - \gamma \epsilon_Z.$$

Substituting the values of C_1 and C_2 into the displacement formula (23), gives

$$V = \frac{1}{r}\ \frac{1+\gamma}{1-\gamma} \int_b^r \propto Tr\ dr + \frac{r(1+\gamma)(1-2\gamma)}{E} \left(\frac{b^2 pi - c^2 po}{c^2 - b^2} \right) + r \left(\frac{1+\gamma}{1-\gamma} \right) \frac{(1-2\gamma)}{c^2 - b^2} \int_b^c \propto Tr\ dr - r\gamma\epsilon_Z + \frac{b^2 c^2}{r(c^2 - b^2)} \left(\frac{(pi-po)(1+\gamma)}{E} \right) + \left(\frac{b^2}{r(c^2 - b^2)} \right) \frac{1+\gamma}{1-\gamma} \int_b^c \propto Tr\ dr$$

or

$$V = (1+\gamma)\left[\frac{1}{r(1-\gamma)} \int_b^r \alpha T r \, dr + \frac{1}{(1-\gamma)(c^2-b^2)} \left(r(1-2\gamma) + \frac{b^2}{r} \right) \times \int_b^c \alpha T r \, dr + \frac{\gamma(1-2\gamma)}{E} \left(\frac{b^2 p_i - c^2 p_o}{c^2-b^2} \right) + \left(\frac{b^2 c^2}{r(c^2-b^2)} \right) \frac{p_i - p_o}{E} - \frac{r \gamma \epsilon_z}{1+\gamma} \right] . \tag{28}$$

Equation (28) is the general expression for the radial displacement at any distance r and becomes Δb when evaluated at $r = b$. In order to determine the radial displacement by use of equation (28) it is necessary to evaluate the constant strain, ϵ_z. This will be done from the expression for σ_z appearing in equation (c) by substituting $\epsilon_r = \frac{dV}{dr}$, $\epsilon_t = \frac{V}{r}$, and using equation (28). We may calculate the magnitude of ϵ_z from the condition that the sum of the normal forces over the cross-section of the cylinder must equal the external force on the end of the bearing due to end pressure.

The normal force is then

$$\text{Normal force} = \int_b^c \sigma_z \cdot \text{area} = \int_b^c \sigma_z \cdot 2\pi r \, dr$$

$$= 2\pi \int_b^c \sigma_z r \, dr = -p_{end} (c^2-b^2)\pi$$

or

$$\int_b^c \sigma_z r \, dr = - \frac{p_{end}}{2} (c^2-b^2) . \tag{29}$$

Substituting for ϵ_r & ϵ_t in equation (c) gives

$$\sigma_z = \frac{E}{1+\gamma}\left[\epsilon_z + \frac{\gamma}{1-2\gamma}\left(\epsilon_z + \frac{dV}{dr} + \frac{V}{r}\right)\right] - \frac{\alpha TE}{1-2\gamma} .$$

Further substitution for V and $\frac{dV}{dr}$ from equation (23) gives

$$\begin{aligned}\sigma_z = E\Bigg[& \frac{\epsilon_z}{1+\gamma} + \frac{\gamma}{1-2\gamma}\Bigg\{\frac{\epsilon_z}{1+\gamma} + \frac{\alpha T}{1-\gamma} - \frac{1}{r^2(1-\gamma)}\int_b^r \alpha Tr\,dr + \\ & \frac{1}{(1-\gamma)(c^2-b^2)}\left((1-2\gamma)\int_b^c \alpha Tr\,dr - \frac{b^2}{r^2}\int_b^c \alpha Tr\,dr\right) + \\ & \frac{(1-2\gamma)}{E}\left(\frac{b^2pi-c^2po}{c^2-b^2}\right) - \frac{b^2c^2}{r^2(c^2-b^2)}\left(\frac{pi-po}{E}\right) - \frac{\gamma\epsilon_z}{1+\gamma} + \\ & \frac{1}{r^2(1-\gamma)}\int_b^r \alpha Tr\,dr + \frac{1}{(1-\gamma)(c^2-b^2)}\left((1-2\gamma) + \frac{b^2}{r^2}\right)\int_b^c \alpha Tr\,dr + \\ & \frac{(1-2\gamma)}{E}\left(\frac{b^2pi-c^2po}{c^2-b^2}\right) + \frac{b^2c^2}{r^2(c^2-b^2)}\left(\frac{P_i-P_o}{E}\right) - \frac{\gamma\epsilon_z}{1+\gamma}\Bigg\}\Bigg] - \frac{\alpha TE}{1-2\gamma}\end{aligned}$$

Upon simplification, we have

$$\begin{aligned}\sigma_z = E\Bigg[& \frac{2\gamma}{(1-\gamma)(c^2-b^2)}\int_b^c \alpha Tr\,dr + \frac{2\gamma}{E}\left(\frac{b^2pi-c^2po}{c^2-b^2}\right) + \\ & \frac{\gamma\alpha T}{(1-2\gamma)(1-\gamma)} - \frac{\alpha T}{(1-2\gamma)} + \frac{1}{(1+\gamma)}\left(1 + \frac{\gamma}{1-2\gamma} - \right. \\ & \left.\frac{2\gamma^2}{1-2\gamma}\right)\epsilon_z\Bigg]. \end{aligned} \qquad (30)$$

Substituting equation (30) in (29), we obtain

$$E\int_b^c \left[\frac{2r\gamma}{(1-\gamma)(c^2-b^2)} \int_b^c \alpha\, Tr\, dr + \frac{2r\gamma}{E}\left(\frac{b^2pi-c^2po}{c^2-b^2}\right) + \right.$$

$$\left. \frac{\alpha\gamma\, rT}{(1-2\gamma)(1-\gamma)} - \frac{r\alpha T}{(1-2\gamma)} + r\epsilon_z \right] dr = -\frac{P_{end}}{2}(c^2-b^2). \qquad (31)$$

Assuming the temperature gradient is linear, then

$$T = t_i\left(1-\frac{r-b}{c-b}\right)$$

where t_i is the inside temperature and the outer surface temperature is zero.

Evaluating the integral within the integral, we have

$$\alpha t_i\int_b^c r\,dr - \frac{\alpha t_i}{c-b}\int_b^c (r-b)r\,dr = \frac{\alpha t_i}{2}(c^2-b^2) -$$

$$\frac{\alpha t_i}{3}\,\frac{(c^3-b^3)}{c-b} + \frac{\alpha t_i\, b(c^2-b^2)}{2(c-b)} \quad .$$

Integrating equation (31) gives

$$E\left[\frac{\gamma\alpha t_i(c^2-b^2)}{2(1-\gamma)} - \frac{\gamma\alpha t_i(c^3-b^3)}{3(1-\gamma)(c-b)} + \frac{\gamma\alpha t_i\, b(c^2-b^2)}{2(1-\gamma)(c-b)} + \right.$$

$$\frac{\gamma(b^2p_i-c^2p_o)}{E} + \frac{\gamma\alpha t_i}{(1-2\gamma)(1-\gamma)} \left\{\frac{c^2-b^2}{2} - \right.$$

$$\left.\frac{1}{c-b}\left(\frac{c^3-b^3}{3} - \frac{b(c^2-b^2)}{2}\right)\right\} - \frac{\alpha t_i}{1-2\gamma}\left(\frac{c^2-b^2}{2} - \right.$$

$$\left.\left.\frac{c^3-b^3}{3(c-b)} + \frac{b(c^2-b^2)}{2(c-b)}\right) + \frac{\epsilon_z}{2}(c^2-b^2)\right] = -\frac{P_{end}}{2}(c^2-b^2)$$

or

$$\epsilon_z = -\frac{p_{end}}{E} + \frac{\gamma\alpha t_i}{1-\gamma}\left[-1 + \frac{2}{3}\frac{(c^2+cb+b^2)}{(c^2-b^2)} - \frac{b}{c-b}\right] -$$

$$\frac{2\gamma(b^2p_i-c^2p_o)}{E(c^2-b^2)} - \left(\frac{\gamma\alpha t_i}{(1-2\gamma)(1-\gamma)}\right) \tag{32}$$

$$\left[1 - \frac{1}{c-b}\left(\frac{2(c^3-b^3)}{3(c^2-b^2)} - b\right)\right] + \frac{\alpha t_i}{1-2\gamma}\left[1 - \frac{2(c^2+cb+b^2}{3(c^2-b^2)} + \frac{b}{c-b}\right].$$

Equation (32) above is a general expression for ϵ_z. Numerical values may be substituted in this equation and the value of ϵ_z, so obtained, may then be substituted in equation (28) for the radial displacement. The radial displacement includes the effects of a temperature gradient across the bearing and the effects of pressure on the bearing surfaces.

The total radial displacement is obtained by superimposing the values of V obtained by equation (28) and that obtained by a uniform temperature change throughout the bearing. This temperature change is equal to the change in the temperature of the outside surface of the bearing. This uniform heating throughout the bearing does not impart any stress so that the method of superposition is valid.

APPENDIX III

DATA SHEET and CALCULATED DATA

RUN NO.	1	2	3	4	5	6	7
RPM	0	0	0	0	0	0	0
p_1	190	110	190	233	225	380	500
p_2	102	103	138	160	160	255	263
$p_3 = p_4$	51	51	51	50	50	51	50
FLOW (Lb.)	10	25	10	25	25	25	25
TIME (MIN-SEC) TOP BRG	2-52.0	6-19.0	1-51.2	3-27.0	3-17.0	2-15.8	1-20.8
TIME (MIN-SEC) BOT. BRG	2-13.0	5-45.5	1-21.2	3-09.5	3-07.0	1-51.2	1-01.0
t_1	84	92	82	89	99	92	201
t_2	84	92	82	89	96	92	200
t_3	85	92	83	89	—	92	201
t_4	83	92	84	89	97	92	200
t_5	84	92	83	89	98	92	201
t_6	84	92	83	—	—	92	201
t_7	83	92	81	89	98	92	201
t_8	84	92	82	89	81	92	201
t_9	84	93	83	89	99	92	201
t_{10}	81	90	80	87	94	91	185
t_{11}	81	90	78	87	94	92	189
t_{12}	81	90	78	87	94	92	188
t_{13}	82	91	82	89	98	92	200
t_{14}	84	91	82	83	98	92	200
t_{15}	87	77	77	76	80	78	84
t_{16}	84	—	82	—	98	—	200
$p_2 - p_4$	51	52	87	110	110	204	213
$t_7+t_3/2$	84	92	82	89	98	92	201
R TOP BRG	80	100.	121	177	206	278	1158
R BOT BRG	—	110	—	193	218	341	—
$m^3P/\nu^2\rho l$	141	175	232	344	422	676	4160
$r m \omega/\nu$	0	0	0	0	0	0	0

DATA SHEET and CALCULATED DATA

RUN NO.	8	9	10	11	12	13	14	
RPM	0	0	0	0	0	0	0	
p_1	770	440	665	1050	970	810	780	
p_2	280	303	438	460	490	490	500	
$p_3 = p_4$	50	47	51	49	50	50	50	
FLOW (Lb.)	25	25	25	25	25	25	25	
TIME (MIN-SEC) TOP BRG	1-55.8	1-59.0	1-29.0	0-56.8	1-10.2	1-15.0	1-18.8	
TIME (MIN-SEC) BOT. BRG	1-31.0	1-34.2	1-13.0	0-47.0	0-57.0	1-08.5	1-06.0	
t_1	87	92	96	203	94	101	97	
t_2	87	92	96	202	93	101	97	
t_3	87	92	95	203	94	—	96	
t_4	87	92	96	202	94	100	97	
t_5	87	92	96	204	95	101	97	
t_6	87	92	95	204	95	—	96	
t_7	87	92	96	203	95	101	97	
t_8	87	92	96	204	95	81	97	
t_9	86	92	95	203	95	99	96	
t_{10}	85	91	93	185	88	98	96	
t_{11}	85	92	94	190	88	98	96	
t_{12}	85	92	94	190	87	98	96	
t_{13}	87	92	96	204	96	101	97	
t_{14}	87	78	96	204	96	101	97	
t_{15}	77	78	78	84	77	80	78	
t_{16}	87	—	—	202	90	100	—	
$p_2 - p_4$	230	256	387	411	440	440	450	
$t_7 + t_3 / 2$	87	92	95.5	203	94.5	101	96.5	
$R_{\text{TOP BRG}}$	309	320	446	1670	555	560	508	
$R_{\text{BOT BRG}}$	—	404	543	—	—	611	607	
$m^3 P / \nu^2 \rho l$	679	841	1380	8060	1499	1735	1616	
$r m \omega / \nu$	0	0	0	0	0	0	0	

DATA SHEET and CALCULATED DATA

RUN NO.	15	16	17	18	19	20	21	
RPM	0	0	0	0	0	0	0	
p_1	1335	1195	1800	1575	1400	1500	1635	
p_2	660	680	720	772	840	860	890	
$p_3 = p_4$	50	50	50	50	50	50	50	
FLOW (Lb.)	25	25	25	25	25	25	25	
TIME (MIN-SEC) TOP BRG	0-54.0	0-56.0	0-43.5	0-47.6	0-49.7	0-49.8	0-45.5	
TIME (MIN-SEC) BOT. BRG	0-45.2	0-52.0	0-32.0	0-40.8	0-47.6	0-47.4	0-43.2	
t_1	109	98	207	110	118	104	100	
t_2	107	98	207	110	117	104	100	
t_3	108	98	207	110	117	102	98	
t_4	108	98	207	110	117	104	100	
t_5	109	99	208	110	118	104	100	
t_6	109	97	206	110	—	—	97	
t_7	109	99	208	111	118	104	98	
t_8	109	99	208	111	118	104	98	
t_9	108	97	208	110	116	100	96	
t_{10}	—	95	189	107	110	104	98	
t_{11}	85	96	195	104	109	105	98	
t_{12}	87	96	194	104	109	105	98	
t_{13}	110	99	208	111	118	104	98	
t_{14}	110	99	208	111	118	104	96	
t_{15}	78	78	84	78	81	80	78	
t_{16}	106	—	206	109	114	102	—	
$p_2 - p_4$	610	630	670	722	790	810	840	
$t_7 + t_3/2$	108.5	98.5	207.5	110.5	117.5	103	98	
R TOP BRG	846	732	2230	977	1000	865	893	
R BOT BRG	—	786	—	—	1050	910	942	
$ma^3P/\nu^2\rho l$	2790	2320	13,230	3400	4240	3230	2990	
$rm\omega/\nu$	0	0	0	0	0	0	0	

DATA SHEET and CALCULATED DATA

RUN NO.	22	23	24	25	26	27	28	
RPM	0	0	0	4000	3500	4000	3500	
p_1	2100	1500	1550	224	210	223	215	
p_2	1000	1120	1170	100	101	148	150	
$p_3 = p_4$	50	50	50	50	49	50	50	
FLOW (Lb.)	25	25 TOP	25 TOP	15	10	15	10	
TIME (MIN-SEC) TOP BRG	0-39.6	0-35.0	0-36.0	6-09.5	2-53.0	3-05.0	1-29.5	
TIME (MIN-SEC) BOT. BRG	0-35.0	SECURED	SECURED	3-33.0	1-56.2	1-51.0	1-03.5	
t_1	108	205	207	96	106	93	107	
t_2	108	204	206	96	106	92	107	
t_3	106	203	205	96	105	93	107	
t_4	108	194	195	95	106	92	107	
t_5	108	199	200	95	106	92	107	
t_6	106	204	205	95	105	93	107	
t_7	108	206	207	96	106	93	107	
t_8	108	181	184	95	106	92	107	
t_9	106	181	184	95	105	93	106	
t_{10}	106	179	180	92	105	87	105	
t_{11}	106	194	192	92	105	85	105	
t_{12}	106	194	192	92	105	86	105	
t_{13}	108	—	—	95	105	92	107	
t_{14}	108	208	207	95	105	93	107	
t_{15}	78	84	84	73	78	71	78	
t_{16}	105	203	204	96	104	93	106	
$p_2 - p_4$	950	1070	1120	50	52	98	100	
$t_7+t_3/2$	107	204.5	206	96	105.5	93	107	
$R_{\text{TOP BRG}}$	1142	2720	2580	64	103	131	200	
$R_{\text{BOT BRG}}$	—	0	0	111	—	214	—	
$m^3P/\nu^2\rho l$	4130	19,840	21,030	180	243	345	466	
$r m\omega/\nu$	0	0	0	440	438	435	437	

DATA SHEET and CALCULATED DATA

RUN NO.	29	30	31	32	33	34	35	
RPM	4000	3800	4000	2560	2780	3860	3800	
p_1	450	625	765	875	1110	1050	1180	
p_2	280	340	450	450	565	585	600	
$p_3 = p_4$	50	50	50	50	50	50	50	
FLOW (Lb.)	25	25	25	25	25	25	25	
TIME (MIN-SEC) TOP BRG	2-03.0	1-36.3	1-19.0	1-03.5	0-54.2	1-01.6	1-00.5	
TIME (MIN-SEC) BOT. BRG	1-37.8	1-21.0	1-07.5	0-55.5	0-49.0	0-55.2	0-51.8	
t_1	95	99	97	142	138	99	102	
t_2	95	99	97	139	138	98	101	
t_3	95	98	96	141	138	98	101	
t_4	95	99	96	140	138	98	101	
t_5	95	99	96	141	138	98	101	
t_6	95	98	96	142	138	97	101	
t_7	95	99	97	142	138	99	102	
t_8	95	99	97	142	138	99	102	
t_9	94	100	95	142	—	96	109	
t_{10}	93	100	93	131	131	95	98	
t_{11}	93	99	93	132	133	96	98	
t_{12}	93	99	93	131	131	96	97	
t_{13}	95	99	96	142	136	99	102	
t_{14}	95	99	97	142	136	99	102	
t_{15}	75	78	76	84	84	77	83	
t_{16}	94	98	95	140	134	97	100	
$p_2 - p_4$	230	290	400	400	515	535	550	
$t_7 + t_3/2$	95	98.5	96.5	141.5	138	98.5	101.5	
R TOP BRG	319	460	507	970	1102	666	633	
R BOT BRG	400	—	592	—	—	742	—	
$m^3 P/\nu^2 \rho l$	810	1300	1436	3380	4050	1978	2060	
$r m \omega/\nu$	437	455	442	431	452	435	429	

DATA SHEET and CALCULATED DATA

RUN NO.	36	37	38	39	40	41	42
RPM	2740	3740	3600	7200	7500	7200	8000
p_1	1650	1550	1600	100	205	200	200
p_2	800	820	820	79	152	153	147
$p_3 = p_4$	50	50	50	18	50	50	32
FLOW (Lb.)	25	25	25	10	10	15	25
TIME (MIN-SEC) TOP BRG	0-42.5	0-47.6	0-47.0	4-18.5	2-21.0	3-28.5	6-08.0
TIME (MIN-SEC) BOT. BRG	0-37.0	0-41.3	0-42.0	2-59.2	1-43.8	2-27.0	4-26.0
t_1	138	100	103	118	120	119	115
t_2	138	100	103	118	120	119	115
t_3	136	99	101	116	120	116	114
t_4	138	100	103	118	118	119	114
t_5	137	100	103	118	120	119	114
t_6	136	99	101	116	120	116	113
t_7	138	100	103	118	120	119	115
t_8	138	100	103	118	120	119	115
t_9	—	97	103	111	120	112	110
t_{10}	135	97	103	112	113	116	109
t_{11}	136	97	103	116	110	120	108
t_{12}	134	97	103	115	111	119	108
t_{13}	137	100	103	116	119	118	114
t_{14}	137	100	103	116	119	118	115
t_{15}	84	78	82	76	82	77	79
t_{16}	138	97	100	114	118	116	112
$p_2 - p_4$	750	770	770	61	102	103	115
$t_7 + t_3 / 2$	138	99.5	102	117	120	117.5	114.5
R TOP BRG	1405	852	905	77	145	144	132
R BOT BRG	—	980	—	111	—	204	182
$m a^3 P / \nu^2 \rho l$	5760	2840	3000	348	616	592	620
$r m \omega / \nu$	443	422	417	995	1065	1000	1076

DATA SHEET and CALCULATED DATA

RUN NO.	43	44	45	46	47	48	49
RPM	8000	8200	8000	8600	8000	8000	9500
p_1	200	525	600	725	1200	1580	205
p_2	147	260	355	405	670	820	102
$p_3 = p_4$	32	50	50	50	50	50	50
FLOW (Lb.)	15	10	25	25	25	25	10
TIME (MIN-SEC) TOP BRG	3-15.0	1-10.0	2-00.8	1-48.1	1-03.5	0-54.0	6-03.6
TIME (MIN-SEC) BOT. BRG	2-36.7	0.53.6	1-33.5	1-26.3	0-54.0	0-45.0	3-59.5
t_1	116	108	108	104	108	110	136
t_2	116	108	108	103	108	110	134
t_3	115	108	106	103	107	108	130
t_4	116	108	107	103	108	110	132
t_5	116	108	107	104	108	110	133
t_6	114	108	106	107	107	108	130
t_7	116	109	107	104	108	110	135
t_8	116	109	107	104	108	110	133
t_9	111	—	104	101	105	106	124
t_{10}	111	104	104	99	105	107	125
t_{11}	111	103	107	98	105	106	124
t_{12}	111	103	107	98	105	106	126
t_{13}	115	108	108	104	109	110	131
t_{14}	115	108	108	104	109	110	132
t_{15}	80	82	82	83	84	85	82
t_{16}	113	107	105	101	105	107	128
$p_2 - p_4$	115	210	305	355	620	770	52
$t_7 + t_3 / 2$	115.5	108.5	106.5	103.5	107.5	109	132.5
R TOP BRG	151	247	372	399	708	847	62
R BOT BRG	188	—	477	—	835	1012	—
$m^3 P / \nu^2 \rho l$	692	900	1380	1460	2770	3470	386
$r m \omega / \nu$	1090	977	988	959	987	995	1490

DATA SHEET and CALCULATED DATA

RUN NO.	50	51	52	53	54	55	56
RPM	9800	10,600	12,000	10,800	12,000	12,000	12,000
p_1	205	700	755	1175	1145	1400	1620
p_2	151	400	450	600	605	780	850
$p_3 = p_4$	50	51	50	50	50	50	50
FLOW (Lb.)	10	25	25	25	25	25	25
TIME (MIN-SEC) TOP BRG	2-46.0	2-05.2	2-06.2	1-20.0	1-22.0	1-05.5	1-00.4
TIME (MIN-SEC) BOT. BRG	2-04.0	1-35.2	1-35.0	1-04.0	1-04.5	0-52.0	0-49.6
t_1	131	121	108	116	106	105	104
t_2	130	122	108	116	105	105	104
t_3	129	118	104	112	112	102	100
t_4	129	121	107	116	105	105	104
t_5	130	121	108	116	106	105	104
t_6	129	118	105	113	103	102	101
t_7	131	121	108	116	106	105	104
t_8	130	121	108	116	106	105	104
t_9	123	117	100	114	99	98	—
t_{10}	121	119	102	115	101	101	105
t_{11}	118	120	102	114	101	101	109
t_{12}	119	120	102	114	101	101	108
t_{13}	129	121	108	116	106	105	105
t_{14}	130	121	108	116	106	105	105
t_{15}	82	82	78	83	78	78	83
t_{16}	126	116	102	115	100	100	99
$p_2 - p_4$	101	349	400	550	555	730	800
$t_7 + t_3 / 2$	130	119.5	106	114	104	103.5	102
R TOP BRG	135	412	353	612	533	664	721
R BOT BRG	—	—	469	—	676	833	—
$m^3 P / v^2 \rho l$	733	2090	1780	2920	2350	2990	3260
$r m \omega / v$	1525	1503	1475	1452	1442	1421	1420

DATA SHEET and CALCULATED DATA

RUN NO.	57	58	59	60	61	62	63
RPM	12,000	11,100	8200	11,100	8800	8200	8200
p_1	1820	205	200	205	200	425	665
p_2	955	100	105	149	152	258	380
$p_3 = p_4$	50	51	51	51	50	51	50
FLOW (Lb.)	25	10	5	10	15	20	25
TIME (MIN-SEC) TOP BRG	0-50.0	5-42.0	1-47.5	2-46.2	2-55.0	1-40.5	1-24.2
TIME (MIN-SEC) BOT. BRG	0-43.0	2-59.6	1-15.0	2-05.5	2-12.0	1-19.5	1-10.2
t_1	105	152	196	149	189	196	198
t_2	104	150	195	147	187	195	197
t_3	101	144	195	144	189	195	198
t_4	105	148	194	145	185	195	197
t_5	104	150	194	146	187	196	198
t_6	102	145	195	144	188	195	197
t_7	105	152	196	149	188	196	198
t_8	105	150	194	146	186	196	198
t_9	98	138	194	133	188	194	197
t_{10}	101	139	179	133	168	180	182
t_{11}	101	138	188	131	170	186	186
t_{12}	101	139	186	134	172	187	188
t_{13}	105	147	190	145	180	194	198
t_{14}	105	148	187	147	186	195	198
t_{15}	79	83	78	84	78	78	80
t_{16}	100	141	194	140	188	194	196
$p_2 - p_4$	905	49	54	98	102	207	330
$t_7 + t_3 / 2$	103	148	194.5	146.5	188.5	195.5	198
$R_{\text{TOP BRG}}$	868	77.	168	156	294	723	1088
$R_{\text{BOT BRG}}$	1010	—	242	—	391	913	1309
$m^3 P/2 \nu^2 \rho l$	3640	490	992	942	1691	3770	6060
$r\,m(a)/2$	1413	2020	2010	2140	2050	2010	2015

DATA SHEET and CALCULATED DATA

RUN NO.	64	65	66	67	68	69	70
RPM	8200	8200	8200	10,400	10,400	10,400	10,400
p_1	685	1150	1615	200	205	475	685
p_2	410	600	850	103	147	273	435
$p_3 = p_4$	50	50	50	51	50	50	49
FLOW (Lb.)	25	25	25	10	15	25	25
TIME (MIN-SEC) TOP BRG	1-21.0	0-57.2	0-44.8	4-12.0	3-03.5	2-04.0	1-19.9
TIME (MIN-SEC) BOT. BRG	1-06.2	0-48.7	0-38.2	3-01.0	2-21.0	1-40.0	1-05.8
t_1	198	199	199	196	196	195	198
t_2	196	197	198	194	195	193	197
t_3	197	197	197	195	195	194	197
t_4	195	198	198	195	195	195	197
t_5	197	198	199	196	196	195	198
t_6	197	198	197	195	195	194	197
t_7	197	198	199	196	196	195	198
t_8	197	198	199	196	196	195	198
t_9	196	195	195	193	192	193	196
t_{10}	179	185	188	183	183	189	183
t_{11}	181	188	188	188	185	192	186
t_{12}	183	188	189	186	184	191	186
t_{13}	196	198	198	—	190	194	198
t_{14}	197	198	198	184	193	194	198
t_{15}	86	85	85	84	84	84	85
t_{16}	195	196	195	192	193	193	196
$p_2 - p_4$	360	550	800	52	97	223	386
$t_7 + t_3 / 2$	197	198	198	195	195.5	194.5	197.5
R TOP BRG	1130	1600	2040	145	296	723	1140
R BOT BRG	1385	1882	2400	199	386	906	1388
$m_1^3 P / \nu^2 \rho l$	6540	9850	14,050	968	1798	4060	7020
$r m \omega / \nu$	2010	1998	1985	2560	2560	2540	2440

DATA SHEET and CALCULATED DATA

RUN NO.	71	72
RPM	10,400	10,400
p_1	1040	1550
p_2	570	835
$p_3 = p_4$	49	51
FLOW (Lb.)	25	25
TIME (MIN-SEC) TOP BRG	1-04.0	0-49:8
TIME (MIN-SEC) BOT. BRG	0-53.0	0-39.8
t_1	192	198
t_2	192	197
t_3	190	195
t_4	192	197
t_5	192	197
t_6	190	196
t_7	192	198
t_8	192	198
t_9	189	197
t_{10}	183	185
t_{11}	187	188
t_{12}	187	187
t_{13}	192	198
t_{14}	192	198
t_{15}	84	85
t_{16}	189	194
$p_2 - p_4$	521	784
$t_7 + t_3/2$	191	196.5
R TOP BRG	1401	1822
R BOT BRG	1698	2290
$m^3 P / v^2 \rho l$	9110	13,680
$r\,m(\omega)/v$	2500	2410